**I. *Multiple Choice*:** Determine if the first statement is true, the second statement is true, both, or neither for:

1. I. If $x$ is a solution to an $n^{th}$ order linear homogeneous differential equation, then so is $1$. T

   II. If $x$ is a solution to an $n^{th}$ order linear homogeneous differential equation, then so is $x^2$.

   a) I only b) II only c) I and II d) neither I nor II. d

2. I. If $S$ is a subset of $T$ and $T$ is linearly independent, then so is $S$.

   II. If $S$ is a subset of $T$ and $S$ is linearly independent, then so is $T$.

   a) I only b) II only c) I and II d) neither I nor II. b

3. I. If $\partial/\partial y\ (f(x,y))$ *and* $f(x,y)$ are polynomials then $dy/dx = f(x,y)$, $f(a) = b$ has a unique solution in some open interval containing a.

   II. If $dy/dx = f(x,y)$, $f(a) = b$ has a unique solution in some open interval containing a, then $\partial/\partial y\ (f(x,y))$ *and* $f(x,y)$ are polynomials. F

   a -4

   a) I only b) II only c) I and II d) neither I nor II.

4. I. If $f_1 = f_2 f_3$, then $\{f_1, f_2, f_3\}$ is linearly dependent.

   II. If $f_1 = f_2 - f_3$, then $\{f_1, f_2, f_3\}$ is linearly dependent.

   a) I only b) II only c) I and II d) neither I nor II. b

5. I. If $x^n + a_{n-1}x + ... + p$ is a polynomial with integer coefficients and $p$ is a prime, then the only possible rational roots are $\pm 1$ *and* $\pm p$.

   II. If the only roots of $x^n + a_{n-1}x + ... + a_0$ are rational, then $a_0$ is an integer.

   a) I only b) II only c) I and II d) neither I nor II.

   a

6. I. If $W(f_1, f_2) = 0$, then $\{f_1, f_2\}$ is linearly dependent. F

II. If $\{f_1, f_2\}$ is linearly dependent and $W(f_1, f_2)$ is defined., then $W(f_1, f_2) = 0$.

a) I only b) II only c) I and II d) neither I nor II.

II. Solve any 5 of the following :

a) $dy/dx = xye^{x^2}$ : $y(0) = 1$

$$\frac{\frac{dy}{dx}}{y(x)} = e^{x^2} x$$

$$\int \frac{\frac{dy}{dx}}{y} dx \int e^{x^2} x dx$$

$$\log(y(x)) = \frac{e^{x^2}}{2} + c_1$$

$$y(x) = c_1 e^{\frac{e^{x^2}}{2}}$$

+ find C

2

b) $xy^2y' = x^2 + 2y^2$

OMIT

what's $\int e^{-3x^2/2}\,dx$ ?

c) $y' + xy = y^4$

$$y = \frac{\sqrt[3]{-2}}{\sqrt{C_1 e^{3x^2/2} - \sqrt{6\pi}\, e^{\frac{3x^2}{2}} \left(\sqrt{\tfrac{3}{2}}\, x\right)}}$$

Div both sides by $-\frac{1}{3} y^4$

$$-\frac{3\frac{dy}{dx}}{y(x)^4} - \frac{3x}{y^3} = -3$$

$$v(x) = \frac{1}{y^3} = \frac{3\frac{dy}{dx}}{y^4}$$

$$u = e^{\int -3x\,dx} = e^{-3x^2/2}$$

sub $-3e^{-3x^2/2}$ $\quad x = \frac{d}{dx}\left(e^{-\frac{3x^2}{2}}\right)$

$$v(x) = -\frac{3}{e^{\frac{3x^2}{2}}}$$

$$\int \frac{d}{dx}\left(\frac{v(x)}{e^{\frac{3x^2}{2}}}\right) dx = \int -\frac{3}{e^{\frac{3x^2}{2}}}\,dx$$

$$v(x) = e^{\frac{3x^2}{2}}\left(-\left(\sqrt{\frac{3\pi}{2}}\left(\sqrt{\tfrac{3}{2}}\,x\right) + C\right)\right)$$

$$y(x) = \frac{(-1)^{2/3}\sqrt[3]{2}}{\left(-\sqrt{6\pi}\, e^{\frac{3x^2}{2}}\right)\left(\sqrt{\tfrac{3}{2}}\, x\right) + C_1 e^{\frac{3x^2}{2}}\right)^{1/3}}$$

-1/2

-2

d) $(2x + y)y' = -2y - 4x$

$$2x\,y'(x) + y(x)\,y'(x) = -2y - 4x$$

$$= -2y - 4x$$

$$(y(x) + 2x)\frac{dy}{dx} = -2y - 4x \wedge \quad \frac{dy}{dx} = -2$$

$$y(x) = \int -2\,dx = -2x + c_1$$

$$y(x) = C_1$$

$$y(x) = -2x$$

$$y(x) = C_1 - 2x$$

e) $y' = (x + y)^{1/2}$

$$v(x) = y(x) + x \quad \frac{dv(x)}{dx} = \frac{dy}{dx} + 1$$

$$\frac{dv}{dx} - 1 = \sqrt{v(x)}$$

$$\frac{\frac{dv}{dx}}{\sqrt{v(x)} + 1} = 1$$

$$\int \frac{\frac{dv}{dx}}{\sqrt{v(x)} + 1}\,dx = \int 1\,dx$$

HOW?

-1 1/2

$$-2\log\left(\sqrt{v(x)} + 1\right) + 2\sqrt{v(x)} = x + C_1$$

$$v(x) = \left(W\left(-e^{\frac{1}{2}(-x-C-2)}\right) + 1\right)^2$$

$$y(x) = -x + \left(W\left(-e^{\frac{1}{2}(-x+C)}\right) + 1\right)^2$$

f) $y'' = (y')^2$.

$$\frac{d^2(y(x))}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

$$\frac{dv}{dx} = v^2 \qquad \frac{\frac{dv}{dx}}{v^2} = 1$$

$$\int \frac{\frac{dv}{dx}}{v(x)^2}\,dx = \int 1\,dx$$

$$-\frac{1}{v(x)} = x + C \qquad v = \frac{1}{x + C}$$

$$\frac{dy}{dx} = -\frac{1}{x + C}$$

$$y(x) = \int -\frac{1}{x + C}\,dx = -\log(x + C) + C_2$$

OK.

III. a) What does it mean for $\{v_1, v_2, ..., v_n\}$ to be linearly independent?

If $c_1v_1 + c_2v_2 + \dots\dots c_nv_n = 0$

then $c_1 = c_2 = c_n = 0$

is the only soln

ok.

b) Which of the following are independent? Which are not? Explain!

i) $S_1 = \{sin(x),\ cos(x)\ ,\ 0\}$

$c_1 \sin x + c_2 \cos x + c_3 0$

$c_1 = 0 \quad c_2 = 0$

$c_1 \sin x = 0$

$c_1 = 0$

linearly independent

not multiple of each other.

-5

It's ~~in~~ dependent!

ii) $S_2 = \{e^x,\ xe^x\ ,\ e^{2x}\}$

$c_1 e^x + c_2 x e^x + c_3 e^{2x}$

$x = 0 \quad c_2 + c_3 = 0$

$x = 1 \quad = 0$

$x = -1 \quad = 0$

Dependant

-4½

It's indep.

iii) $S_3 = \{\sin x, \cos x, e^x\}$.

independent

not multiple of each other

(Not for this reason).

-4

IV. Give a differential equations whose solution is $span\ S_i$ $for\ i = 1, 2,\ and\ 3$ in III.
(Recall $span\ \{v_1, v_2, ..., v_n\} = \{c_1 v_1 + c_2 v_2 + ... + c_n v_n : c_i \in R\}$.
*See the board if you don't like this notation*).

i) Give D.E whose Solution is $C_1 \sin x + C_2 \cos x + C_3 0$

$r = \pm i$

$r^2 = -1$

$r^2 + 1 = 0$

$y'' + y = 0$

ii) Give Differential Eq whose soln is $C_1 e^x + C_2 x e^x + C_3 e^{2x}$

$= C_1 e^{\ln x} + C_2 e^x \quad e^{\ln x} = x$

Roots of auxil eq are $m = \ln(x), 1$

$m^2 - (1 + \ln x) m + \ln x = 0$

$y'' - (1 + \ln(x)) y' + (\ln x) y = 0$

-5. Char eqn is $(r-1)^2(r-2) = 0$

$(r^2 - 2r + 1)(r-2) = 0$

iii) Give D.E. whose solution is $C_1 \sin x + C_2 \cos x + C_3 e^x$

$y = \left(\frac{1}{2} + \frac{i}{2}\right) C_1 e^{-ix} + \left(\frac{1}{2} - \frac{i}{2}\right) c e^{ix} + c e^x$

-5

$(r^2+1)(r-1) = 0$ is char eqn.

$r^3 - r^2 + r - 1 = 0$

$\Rightarrow y^{(3)} - y^{(2)} + y' - y = 0$ is de!

V. Solve $y'' - y' - 6y = f(x)$ for

a) $f(x) = 0$

$\frac{dy}{dx} + \frac{d^2y}{dx^2} - 6y = 0$

$y(x) = e^{\lambda x}$ $\frac{d^2}{dx^2}e^{\lambda x} + \frac{d}{dx}e^{\lambda x} - 6e^{\lambda x} = 0$

Sub $\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 6e^{\lambda x} = 0$

factor $(\lambda^2 + \lambda - 6)e^{\lambda x} = 0$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda - 2)(\lambda + 3) = 0$

Roots are $\lambda = -3$ or $2$

$y(x) = y_1 + y_2 = Ce^{-3x} + Ce^{2x}$

b) $f(x) = e^x$

$\frac{dy}{dx} + \frac{d^2y}{dx^2} - 6y = e^x$

$y(x) = e^{\lambda x}$ sub

$\frac{d^2}{dx^2}(e^{\lambda x}) - 6e^{\lambda x} - (\frac{d}{dx})e^{\lambda x} = 0$

$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 6e^{\lambda x} = 0$

$(\lambda^2 + \lambda - 6)e^{\lambda x} = 0$ $(\lambda - 2)(\lambda + 3)$

$\lambda = -3$ or $2$

$y_p(x) = a_1 e^x$

$a_1 e^x + a_1 e^x - 6(a_1 e^x) = e^x$ $-1/2$

$-4a e^x = e^x$ $-4a = 1$ $a = -\frac{1}{4}$

$y_p = -e^x/4$

$y(x) = -\frac{e^x}{4} + ce^{-3x} + C_2 e^{2x}$

c) $f(x) = x$

$\frac{d^2y}{dx^2} + dy/dx - 6y = 0$ $y(x) = e^{\lambda x}$

$(\lambda^2 + \lambda - 6)e^{\lambda x} = 0$

$\lambda^2 + \lambda - 6 = 0$ $\lambda = -3$ or $2$

$y_p = a_1 + a_2 x$

$\frac{dy_p(x)}{dx} = \frac{d}{dx}(a_1 + a_2 x) = a_2$

$\frac{d^2y_p}{dx^2} + \frac{dy_p}{dx} - 6y_p = x$

$a_2 - 6(a_1 + a_2 x) = x$

$(-6a_1 + a_2) - 6a_2 x = x$

$-6a_2 = 1$

$a_1 = -\frac{1}{36}$ $a_2 = \frac{-1}{6}$

$y(x) = \frac{-x}{6} + C_1 e^{-3x} + C_2 e^{2x} + \frac{1}{36}$

VI. Solve :

a) $y^{(4)} + 2y'' + y = 0$

$f_1(y(x)) = 1$ $f_2(\frac{d}{dx}y(x)) = 2$

$g_1(y(x)) = -y^4(x) - y(x)$

$g_2(\frac{d}{dx}y(x)) = 1$

$\leftarrow y' = dy/dx$

$2dx\frac{d^2}{dx^2}y(x) = dx(-y^4(x) - y(x))$

$\int 2\frac{d}{dx}y \frac{d^2}{dx^2}y\,dx = \frac{d}{dx}y(x)^2$

$\frac{d}{dx}y^2 = C_1 - \frac{1}{5}y^5 - \frac{1}{2}y^2$

$\frac{d}{dx}y = -\frac{1}{10}\sqrt{100C_1 - 20y^5(x) - 50y^2(x)}$

$x = C_2 + \sqrt{10}\int \frac{1}{\sqrt{10C_1 - 2y^5 - 5y^2}}dy$

~~Let~~

Just solve.

$r^4 + 2r^2 + 1 = 0$.

$-4\frac{1}{2}$

24

b) $y^{(3)} - 3y'' - 4y' + 12y = 0$

$r^3 - 3r^2 - 4r + 12 = 0$

$f(1) = r^3 - 3r^2 - 4 + 12 = 0$

$f(3) = 27 - 27 - 12 + 12 = 0$

$$\begin{array}{r|rrrr} 3 & 1 & -3 & -4 & -12 \\ & & 3 & 0 & 12 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

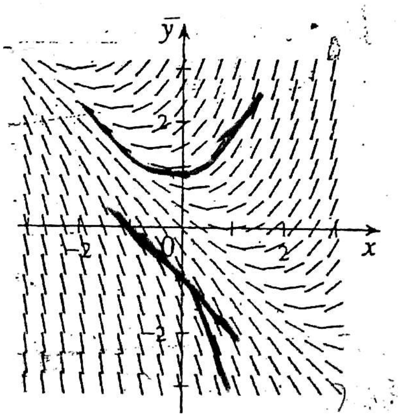
-1/2

$r^2 + 4 = 0$

$r = \pm\sqrt{2}\, i$

$y = C_1 e^{3x} + C_2 \cos\sqrt{2}x + C_3 \sin\sqrt{2}x$

VII. Sketch the solution curves with $y(0) = 1$ *and* $y(0) = -1$ for the given slope field:




-1 Follow the cure