These rules are identical to those developed previously and may be applied to a statistical assessment of the difference between two proportions.

Test Concerning Multiple Proportions

Although it is useful to examine the difference between two proportions, the health administrator is frequently required to evaluate differences among multiple proportions. In the previous section, we identified two subgroups, males and females, as the focus of analysis. However, it is possible to partition a population into three or more groups. For example, we might be interested in employee satisfaction, grouped by occupational categories such as physicians, nurses, technicians, and support staff. Similarly, we might be interested in the evaluation of care by individuals of different socioeconomic groups or variation in the use of care by individuals whose service is financed by different types of payment. In each of these cases, three or more subgroups are of interest to the analyst.

In general, we presume that the population of interest consists of k subgroups and classify the number of observations in each in terms of the presence or absence of the attribute of interest. For example, we might be interested in patients whose discharge from the hospital to a long-term care facility was delayed versus those individuals who experienced a timely transfer. Thus, the focus is on the presence or absence of a delay in transferring the patient to a source of care after the hospital episode. We also might group patients into k categories that depict the racial status of the patient or the individual's source of insurance coverage.

After the k categories are defined, the analyst can construct a table consisting of two rows, defined by the presence or absence of a delay in discharge, and k columns, each of which corresponds to one of several ethnic backgrounds or one of several sources of insurance coverage. After determining the number of patients or observations in each cell of the table, the analyst is in a position to determine whether the observed distribution of observations is compatible or consistent with a postulated theory or a presumption that is specified by the null hypothesis.

For example, suppose that the administrator is interested in employees' satisfaction with their conditions of employment. Assume that a survey was administered to employees and that in addition to a set of questions that addressed specific areas of their employment nurses, technicians, and support staff were asked to respond to the following question:

In general, are you satisfied with the conditions of your employment?

Yes ____ No ____

Table 7.1 summarizes the actual responses of 70 nurses, 50 technicians, and 80 support personnel who were included in the survey.

Table 7.1 **Satisfaction or Dissatisfaction with Conditions of Employment**

	Nurses	Technicians	Support Staff
Satisfied	35 (24.5)	20 (17.5)	15 (28)
Dissatisfied	35 (45.5)	30 (32.5)	65 (52.0)
Total	70	50	80

Fifty percent (i.e., 35/70 × 100) of the nurses and 40% (i.e. 20/50 × 100) of the technicians were satisfied with the general conditions of their employment. However, only 18.75% (i.e., 15/80 × 100) of the support staff expressed satisfaction with working conditions. In this case, the object is to determine whether the observed differences in the proportions or percentages of employees reporting a general satisfaction are significant or attributable to chance variation.

The first step is to specify the null and alternate hypotheses. In the following, let π_1 correspond to the proportion of nurses who expressed satisfaction with working conditions and π_2 represent the proportion of technicians who were similarly satisfied. Finally, let π_3 correspond to the proportion of the support staff that reported general satisfaction with their conditions of employment. The null hypothesis, which assumes that the three proportions are identical, might be specified as follows:

$$H_o: \pi_1 = \pi_2 = \pi_3$$

In contrast, the alternate hypothesis assumes that the three proportions are not equal. We specify the alternate hypothesis as

$$H_a: \pi_1 \neq \pi_2 \neq \pi_3$$

As in the examination of the difference between two proportions, we assume that the null hypothesis is true and that π_1, π_2, and π_3 are equal to a common proportion represented by P*. As discussed in the previous section, P* is defined by

$$P^* = (x_1 + x_2 + x_3)/(n_1 + n_2 + n_3)$$

We let x_1 represent the number of nurses who expressed satisfaction with the working environment whereas the number of technicians and support staff reporting satisfaction with their conditions of employment is represented by x_2 and x_3, respectively. Finally, the number of nurses, technicians and staff members responding to the survey is indicated by n_1, n_2 and n_3, respectively. Thus, the common proportion, P*, is obtained by

$$P^* = (35 + 20 + 15)/(70 + 50 + 80)$$

or 0.35. The complement of P*, 1 − P*, is 0.65.

We now presume that the null hypothesis is correct and that the proportions π_1, π_2, and π_3 are estimated by the numeric value of the common proportion P* or 0.35. If the null hypothesis were true, we would expect 35% of each group to express satisfaction with their working conditions and 65% to report dissatisfaction with the work environment. Based on these data, the expected cell frequencies depicting members of each group who are satisfied with the terms of their employment are calculated as follows:

Group	Calculations	Number Satisfied
Nurses	0.35 (70)	24.5
Technicians	0.35 (50)	17.5
Support staff	0.35 (80)	28.0

These calculations suggest that if the null hypothesis is correct 35% of the 70 nurses ought to report that they are satisfied with the terms of their employment and 35% of the 50 technicians ought to report that they also are satisfied with working conditions. The expected number of employees reporting dissatisfaction, grouped by occupational category, is calculated as follows:

Group	Calculations	Number Dissatisfied
Nurses	0.65 (70)	45.5
Technicians	0.65 (50)	32.5
Support staff	0.65 (80)	52.0

The calculations refer to expected cell frequencies that are based on the presumption that the null hypothesis is correct. The expected cell frequencies are summarized in Table 7.1 and appear in parentheses as bold-faced entries.

Differences between the observed frequencies derived from responses to the survey and the expected cell frequencies that are based on the presumption that the null hypothesis is true might be interpreted as follows:

1. If the null hypothesis is true, the absolute value of the differences between the observed and expected cell frequencies will, on balance, be small.
2. If the null hypothesis is false and the alternate hypothesis is true, the absolute value of the differences between the observed and expected cell frequencies will, on balance, be large.

In this case, we require a test statistic and a critical value that enable us to determine whether the differences are large or small.

In the following, let *O* correspond to the observed frequency appearing in a given cell and *e* represent the corresponding expected frequency. Using this notation, we evaluate the differences between the distribution of observed and expected frequencies by relying on the χ^2 distribution, where χ is the Greek letter chi. The test statistic is given by

$$\chi^2 = \Sigma(O - e)^2/e \qquad \text{(Equation 7.13)}$$

The calculated value of χ^2 is obtained by first squaring the difference between the observed and expected frequency and dividing by the expected value for each cell. We then simply sum the results obtained for each cell in the table. The calculated value of χ^2 is interpreted as follows:

1. If the null hypothesis is true, the absolute value of $(O - e)$ will, on balance, be small, suggesting that the ratio $(O - e)^2/e$ will, on balance, be small and that the calculated value of χ^2 will be small.
2. If the null hypothesis is false and the alternate hypothesis is true, the absolute value of $(O - e)$ will be large, suggesting that the ratio $(O - e)^2/e$ will, on balance, be large and that the calculated value of χ^2 will be large.

In Table 7.1, the calculated value of χ^2 is given by

$$\chi^2 = (35 - 24.5)^2/24.5 + (20 - 17.5)^2/17.5 + (15 - 28)^2/28 + (35 - 45.5)^2/45.5 + (30 - 32.5)^2/32.5 + (65 - 52)^2/52$$

or approximately 16.76.

In Exhibit 7.1, an Excel spreadsheet can be used to calculate the value of χ^2. The original or observed frequencies that were derived from the survey are listed in Part A of Exhibit 7.1. The common proportion, P*, is shown in cell B13, whereas its complement is listed in cell B14. Based on the presumption that the null hypothesis is correct, the expected frequencies are shown in Part B of the exhibit and were calculated as follows:

1. Enter = B13*B$10 in cell B23.
2. Use the "COPY" function to record the equation in the other two cells of the first row.
3. Enter = B14*B$10 in cell B24.
4. Use the "COPY" function to record the equation in the other two cells of the second row.

In this case, B14 is a fixed cell address, whereas B$10 limits the calculations to the 10th row of the spreadsheet. The distribution of expected frequencies calculated in Part B of Exhibit 7.1 is identical to the set of results appearing in the set of parentheses shown in Table 7.1.

The process of calculating the value of χ^2 is illustrated in Part C of Exhibit 7.1. The calculated value of χ^2 is obtained by

1. Entering the equation = (B8 − B23)^2/B23 in cell B33
2. Using the "Copy" function to record the remaining cells of Part C
3. Using the "AutoSum" function to calculate row, column, and grand totals

Referring to the equation (B8 − B23)^2/B23, the symbol "^" indicates the operation of exponentiation and results in the square of the difference between the value appearing in cells B8 and B23. The value of χ^2 (16.76) computed in Exhibit 7.1 is identical to the corresponding value listed in Table 7.1.

If the calculated value of χ^2 is large, the differences between the observed and expected frequencies are large. Thus, we are tempted to reject the null hypothesis in favor of the alternate. In contrast, if the calculated value of χ^2 is small, the differences between observed and expected frequencies are also small and probably attributable to chance variation. In such a situation, we are tempted to accept the null hypothesis or reserve judgment.

When evaluating the calculated value of χ^2, we rely on the χ^2 probability distribution. As indicated in Figure 7.2, the χ^2 distribution is unimodal, continuous, and limited to positive values; χ^2 is a family of probability distributions, each determined by a single parameter defined as degrees of freedom. For a relatively

Exhibit 7.1 **Satisfaction or Dissatisfaction with Conditions of Employment**

Part A: The Original Data

Response	**Nurses**	**Technicians**	**Support Staff**	**Total**
Satisfied	35	20	15	70
Dissatisfied	35	30	65	130
Total	70	50	80	200
P*	0.35			
1–P*	0.65			

Part B: Expected Values: Assuming the Null Hypothesis Is Correct

Response	**Nurses**	**Technicians**	**Support Staff**
Satisfied	24.5	17.5	28
Dissatisfied	45.5	32.5	52

Part C: The Calculation of Chi Square

Response	**Nurses**	**Technicians**	**Support Staff**	**Total**
Satisfied	4.50	0.36	6.04	10.89
Dissatisfied	2.42	0.19	3.25	5.87
Total	6.92	0.55	9.29	**16.76**

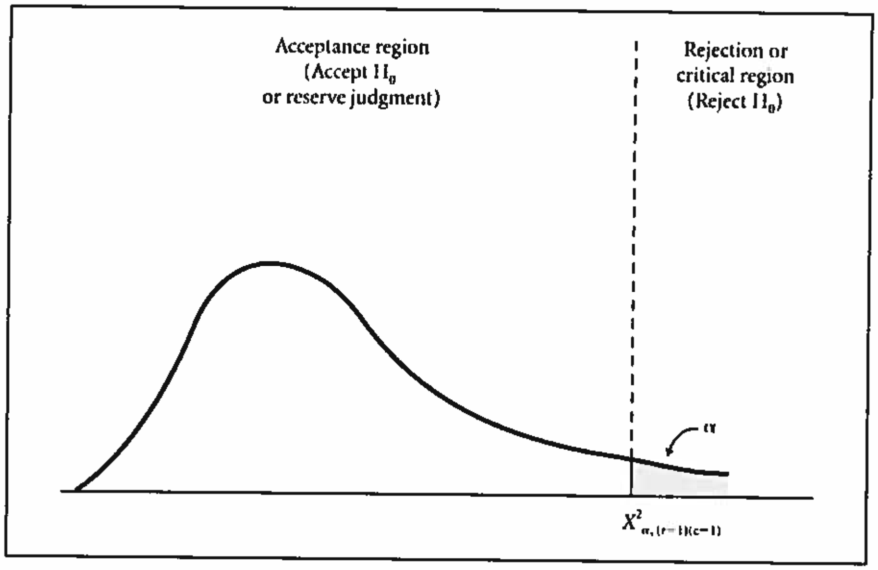

Figure 7.2 **The χ^2 Distribution and the Decision Criteria**

small number of degrees of freedom, the χ^2 distribution is asymmetrical and skewed to the right. However, as the number of degrees of freedom increases, the χ^2 distribution becomes more symmetrical and approaches the normal distribution. When examining the difference between the observed and expected frequencies that appear in a table consisting of r rows and c columns, we see that the number of degrees of freedom is given by the product $(r - 1)(c - 1)$. The data have been organized into a table consisting of two rows and three columns. Thus, we find that $(2 - 1)(3 - 1)$ or 2 degrees of freedom are available for analysis.

The χ^2 probability distribution is in Table D.5. As indicated, the number of degrees of freedom defines the rows of the table, whereas the values of χ^2_α define the columns of the table. Thus, each row of the table corresponds to one of the distributions known as χ^2. The columns of the table are defined for different values of α or $1 - \alpha$. Similar to our discussion of Z_α and t_α, we define $\chi^2_{\alpha,(r-1)(c-1)}$ as the value of chi-square for which the area to its right is α. Figure 7.1 indicates that $\chi^2_{\alpha,(r-1)(c-1)}$ is a critical value that divides the distribution into two regions. Specifically, if the calculated value of χ^2 exceeds the critical value $\chi^2_{\alpha,(r-1)(c-1)}$, we conclude that the differences among the observed and expected frequencies are significant. In this situation, we reject the null hypothesis and accept the alternate, recognizing that the probability of committing a Type I error is α or less. In contrast, if the calculated value of χ^2 is less than the critical value, we conclude that the differences among the expected and observed frequencies are small and not significant. As a consequence, we are forced to accept the null hypothesis or if possible reserve judgment.

Returning to the example, suppose that $\alpha = 0.05$. Given that two degrees of freedom are available, the corresponding value of chi-square, $\chi^2_{0.05,2}$ is 5.991, implying that if the null hypothesis is true the probability of obtaining a value of χ^2 that exceeds 5.991 is 0.05 or less. In Table 7.1 or Exhibit 7.1, the calculated value of χ^2 was 16.76. Because the calculated value of χ^2 exceeds the critical value, we reject the null hypothesis and accept the alternate, recognizing that the probability of committing a Type I error is 0.05 or less.

Viewed from an administrative perspective, the results also indicate that the proportion of employees, grouped by occupational category, expressing satisfaction with their conditions of employment differed significantly and that members of the support staff appear to be the most dissatisfied. Thus, the results indicate the presence of a potential problem or organizational weakness that requires further investigation and perhaps the implementation of a remedial policy.

Tests of Independence or Relationship

The χ^2 distribution is also valuable when the data are arranged in a contingency table and the independence or the association of one variable with another needs to be examined. In general, a contingency table is constructed to assess the relationship of one variable with another or to determine whether the two variables are unrelated or independent. The health administrator might examine two kinds of problems.

The first is a situation in which three or more outcomes are possible while the size of the corresponding samples is known and held constant. For example, suppose that the health service organization is evaluating a proposal to extend the hours of operation in the outpatient clinic and that the opinions of employees will be considered when assessing the suggestion. Assume also that 50 physicians, 70 nurses, and 25 technicians were surveyed and asked to respond to the following question:

In general, do you favor increasing the hours of operation in the clinic by 1 hour?

Favor ____ Oppose ____ Undecided ____

In this case, the size of each sample is fixed and equal to 50 physicians, 70 nurses, and 24 technicians. The outcomes consist of responses that indicate employees who favor the proposal, oppose the proposal, or are undecided. Even though the sample sizes are fixed, responses are allowed to vary and represent the values assumed by the random variable that depicts the opinions of employees included in the survey. In this case, we are interested in examining the null hypothesis that the reaction of employees to the proposal is independent of occupational category. The alternate hypothesis specifies that the reaction of employees is dependent on the type of occupation.

The second kind of problem involves a situation in which sample sizes and outcomes are allowed to vary. For example, suppose that we are interested in any association between the socioeconomic status of patients and their evaluation of the care provided by our organization. Suppose also that we asked 1,000 patients to rate the care that they received as excellent, fair, or poor. After assigning each individual to a social category as low, middle, or high, suppose that we obtained the distribution as shown in Table 7.2.

As before, we are interested in a null hypothesis that assumes that the evaluation of the care provided by our organization is independent of social class. The alternate hypothesis indicates that the evaluation of care is dependent on or related to social class.

To illustrate the approach that is used to assess this situation, we assume initially that the null hypothesis is correct and that the evaluation of care is independent of social class. If the null hypothesis of independence is correct, recall that we may apply the special rule of multiplication, which specifies that

$$P(A \cap B) = P(A)P(B)$$

We now consider the probability of randomly selecting a patient who is of high social class *and* rated the care provided by the organization as excellent. In Table 7.1, A_1, A_2, and A_3 represent an individual of low, middle, and high social class, respectively. Similarly, B_1, B_2, and B_3 identify a response of excellent, fair, or poor to the question. Thus, the desired probability might be expressed as

$$P(A_1 \cap B_1) = P(A_1)P(B_1)$$

After appropriate substitution we find that

$$P(A_1 \cap B_1) = (300/1{,}000)(290/1{,}000)$$

Table 7.2 **Evaluation of Patient Care by Social Class**

	Social Class			
Evaluation	**Low**	**Middle**	**High**	**Total**
	(A_1)	**(A_2)**	**(A_3)**	
Excellent	30	150	110	290
(B_1)	**(87)**	**(145)**	**(58)**	
Fair	120	190	60	370
(B_2)	**(111)**	**(185)**	**(74)**	
Poor	150	160	30	340
(B_3)	**(102)**	**(170)**	**(68)**	
Total	300	500	200	1,000

or 0.087. Furthermore, if the null hypothesis is correct, we would expect (0.087)(1,000) or 87 patients to occupy a high social class *and* rate the care as excellent. If e_{11} corresponds to the expected frequency assigned to the cell defined by the intersection of row one and column one, this result is equivalent to

$$e_{11} = (300/1{,}000)(290/1{,}000)1{,}000$$

or (300)(290)/1,000. These calculations suggest that the expected cell frequency is the product of an appropriate row and column total divided by the total number of observations.

Similarly, we may calculate the probability of selecting a patient who was assigned to the high social class *and* rated the care as poor by

$$P(A_3 \cap B_3) = (200/1{,}000)(340/1{,}000)$$

or 0.068. Hence, if the null hypothesis were correct, we would expect

$$[(200/1{,}000)(340/1{,}000)](1{,}000)$$

or 68 members of the high social class to evaluate the care as poor. Using this approach, the frequency expected in the cell defined by the intersection of the third row and third column is given by

$$e_{33} = (200)(340)/1{,}000$$

or 68 patients.

The calculations summarized here suggest that the expected cell frequency is given by the product of the appropriate row and column totals divided by the total number of observations. In general, let the subscript i correspond to a row in the table and the subscript j represent one of the columns. In addition, let R_i correspond to the total of row i and C_j represent the total of column j. The expected frequency in the cell defined by the intersection of row i and column j is given by

$$e_{ij} = (R_i)(C_j)/\,T \qquad \text{(Equation 7.14)}$$

where T is the total number of observations. After applying Equation 7.14 to each cell of Table 7.2, we obtain the distribution of expected frequencies that appears in bold face and parentheses.

As discussed in the previous section, if differences among the observed and expected frequencies are, on balance, small, we might believe that the null hypothesis is correct (i.e., the social class is independent of the evaluation of care). On the other hand, if the differences among the observed and expected frequencies are large, we tend to believe that the null hypothesis should be rejected in favor of the alternate. Equation 7.13 is used to calculate the value of χ^2 as follows:

$$\chi^2 = (30 - 87)^2/87 + (150 - 145)^2/145 + (110 - 58)^2/58 + (120 - 111)^2/111 + (190 - 185)^2/185 + (60 - 74)^2/74 + (150 - 102)^2/102 + (160 - 170)^2/170 + (30 - 68)^2/68$$

After performing the calculations, we find that the value of χ^2 is approximately 132.06.

The calculations required to determine the value of χ^2 are tedious and may result in errors. However, with Excel, the calculated value of χ^2 is determined with relative ease. In Exhibit 7.2, the original information from the survey was summarized in Part A. The expected cell frequencies were then calculated as follows:

1. Enter the equation = (B$10*$E7)/E10 in cell B18.
2. Use the "COPY" function to record the equation in the remaining cells of Part B.
3. Use the "AutoSum" function to calculate row totals, column totals, and the total number of observations.

The address B$10 restricts calculations to the 10th row of the spreadsheet, whereas $E7 restricts calculations to column E of Exhibit 7.2. Finally, E10 is a fixed cell address in which the total number of observations is recorded.

The method of calculating the value of χ^2 is shown in Part C of Exhibit 7.2. Similar to Exhibit 7.1, the calculated value of χ^2 is obtained by

1. Entering =(B7 − B18)^2/B18 in cell B29
2. Using the "Copy" function to record the equation in other cells forming the body of the exhibit
3. Using the "AutoSum" function to determine row totals, column totals, and the grand total

These calculations indicate that the calculated value of χ^2 is approximately 132.06.

As before, if the calculated value of χ^2 is large, we conclude that the differences among the observed and expected frequencies are large, an outcome that forces us to reject the assumption of independence and conclude that social class is associated with the evaluation of care. On the other hand, if the calculated value of χ^2 is small, the differences among the observed and expected frequencies are small, and we are forced to accept the null hypothesis (i.e., social class and the evaluation of care are independent) or reserve judgment.

From our earlier discussion, it is clear that a comparison of the calculated and critical value of χ^2 forms the basis for evaluating the null and alternate hypothe-

Exhibit 7.2 The Evaluation of Social Class and the Evaluation of Care

Part A: The Original Data

	Social Class			
	Low	**Middle**	**High**	**Total**
EVALUATION				
Excellent	30	150	110	290
Fair	120	190	60	370
Poor	150	160	30	340
Total	300	500	200	1,000

Part B: Expected Cell Frequencies

	Social Class			
	Low	**Middle**	**High**	**Total**
EVALUATION				
Excellent	87	145	58	290
Fair	111	185	74	370
Poor	102	170	68	340
Total	300	500	200	1,000

Part C: The Calculation of Chi-Square

	Social Class			
	Low	**Middle**	**High**	**Total**
EVALUATION				
Excellent	37.34482759	0.172413793	46.62068966	84.13793103
Fair	0.72972973	0.135135135	2.648648649	3.513513514
Poor	22.58823529	0.588235294	21.23529412	44.41176471
Total	60.6627261	0.895784222	70.50463242	132.0632093

sis. If $\alpha = 0.05$, Table D.5 reveals that the critical value of χ^2 for four degrees of freedom is 9.488. Because the calculated value (132.06) greatly exceeds the critical value, we are forced to reject the null hypothesis in favor of the alternate, recognizing that the probability of committing a Type I error is 0.05 or less. Because of magnitude of the calculated value, one might conclude that there is virtually no chance of committing a Type I error.

The analysis clearly indicates that social class and the evaluation of care are associated. An inspection of the observed frequency distribution clearly indicates that those patients assigned to the high socioeconomic group were more likely to rate their care as excellent, whereas those in the low socioeconomic class were more likely to rate their care as poor. As a consequence, the analysis suggests that the treatment of patients of different social groups varies and that further examination or administrative investigation into potential inequities in the services provided by the organization may be required.

Exercises

1. In a sample survey, 200 of 800 individuals enrolled in our managed-care organization indicated that they favored a proposal to extend office hours from 5:00 to 8:00 p.m. Construct a 95% confidence interval for the corresponding true proportion.
2. Suppose that we conducted a survey in which we asked 800 people the following question: "In the event of an emergency, would you seek medical assistance at the emergency room of our hospital?" If 180 of the respondents indicated that they would seek care at the emergency room, construct a 95% confidence interval for the true proportion.
3. In a random sample of 800 older members of our managed-care organization, 260 indicated that they intended to visit one of our clinics for a flu vaccination. Construct a 99% confidence interval for the corresponding true proportion.
4. In a random sample of 1,200 bills submitted to a given insurer, 250 contained an error. Construct a 98% confidence interval for the true proportion of bills that contain an error.
5. Suppose that we believe that no fewer than 60% of members of a prepaid group practice received a physical examination last year. In a sample survey of 800 members, suppose that we found that 352 individuals received an exam during the previous year. If $\alpha = 0.01$, use these results to examine the null hypothesis that the proportion of the membership receiving a physical examination is 0.60.
6. In a random sample of 640 recent discharges, it was found that a hospital-acquired infection was reported in the medical records of 262. We believe that no less than 20% of inpatients should acquire a hospital-related infection. If $\alpha = 0.05$, use these data to evaluate the performance of the hospital.
7. Suppose that an expert tells us that the defective rate (i.e., false positive or false negative) associated with a given laboratory procedure should be no more than 10%. In a random sample of 400 results, suppose that we found that 100 were either a false positive or a false negative. If $\alpha =$ 0.05, use these data to examine the null hypothesis that the probability of a defective result is 0.10.
8. Suppose that we operate two long-term care facilities and are concerned that the quality of care provided by the two facilities differs. In an initial assessment, suppose that we randomly selected 100 patient records from facility A and found that bed sores were