

Question I (70 points) Consider the following modification of the real business cycle model. A representative consumer has preferences over consumption c_t and labour l_t . The expected lifetime utility is defined by

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t l_t^\gamma), \quad \gamma > 1, \quad 0 < \beta < 1.$$

The markets are perfectly competitive. The production function of the representative firm is $y_t = z_t k_t^\alpha l_t^{1-\alpha}$, where k_t is the stock of capital and z_t represents exogenous technology shocks. The stock of capital is owned by the consumer. Assume that capital completely depreciates after one period. That is, the stock of capital evolves as $k_{t+1} = i_t$. The technology shocks follow an autoregressive process of order one:

$$z_t = \rho z_{t-1} + \varepsilon_t, \quad 0 < \rho < 1.$$

The innovations ε_t are independently and identically distributed, $\varepsilon_t \sim N(0, \sigma^2)$. There is no trend in technology.

You can make any additional assumptions, if you find this is necessary. However, in this case, your additional assumptions must be stated explicitly.

1. (10 points) The firm's problem

- (a) (5 points) Write down a firm's optimization problem. Be explicit about which variables the firm chooses and which variables it takes as given.
- (b) (5 points) Derive and interpret the first order necessary conditions of the firm.

2. (15 points) The consumer's problem

- (a) (5 points) Write down the representative consumer's optimization problem. Be explicit about which variables the consumer chooses and which variables (s)he takes as given.
- (b) (10 points) Using the method of Lagrange, or the approach discussed in Romer (2012) "Advanced Macroeconomics," derive the first order necessary conditions for the consumer. Interpret these conditions.

3. **(5 points) Competitive equilibrium.** Define, *in detail*, a competitive equilibrium of this model. Interpret the equilibrium conditions from an economic perspective.

4. **(20 points) Solution**

- (a) (5 points) Make a guess that in equilibrium $c_t = l_t^\gamma = by_t$, ($b > 0$). Show that, if the guess is correct, the equilibrium investment is a constant fraction of output. (*Hint*: you do not need to find the value of b for this subquestion).
- (b) (5 points) Find the value of b , and verify that with this value all the equilibrium conditions of the model are satisfied.
- (c) (5 points) Find the equilibrium employment l_t as a function of the state variables k_t and z_t , and the parameters of the model.
- (d) (5 points) Show that, starting from the initial period $t = 0$, the analytical solution of the model for all endogenous variables can be computed recursively.

5. **(15 points) Implications for the labour market.** Consider the labour market in period t .

- (a) (5 points) Find the expression for labour supply. Plot the labour supply curve. Does the labour supply have a positive or a negative slope? Why? How is the labour supply different from that in the simplified real business cycle model in Romer (2012) “Advanced Macroeconomics”?
- (b) (5 points) Find the expression for labour demand. Plot the labour demand curve. Does the labour demand have a positive or a negative slope? Why?
- (c) (5 points) Suppose the level of technology increases from z_1 to z_2 . Analyze what a positive technology shock has on labour demand and labour supply. Explain the economic intuition behind these effects. What happens to equilibrium employment? Can the model replicate a stylized business cycle fact that employment is procyclical in the actual data? Explain.

6. (5 points) Implications for investment

In the actual data, investment is procyclical and more volatile than output. Can the model replicate these two stylized business cycle facts about investment? Explain your response.

Question 2 (30 points) Consider the consumption-saving model with durable consumption goods. The representative individual receives a random labor income Y_t each period, and maximizes the expected life-time utility by solving the following problem:

$$\max_{\{C_t, D_t, A_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(D_t),$$

subject to

$$A_{t+1} = (1+r)(A_t + Y_t - C_t), \text{ for all } t \quad (1)$$

$$D_t = (1-\delta)D_{t-1} + C_t, \text{ for all } t \quad (2)$$

$$\lim_{t \rightarrow \infty} (1+r)^{-t} A_{t+1} = 0 \quad (3)$$

Here D_t denotes the stock of durables goods, C_t is the consumption expenditures on durable goods, A_t denotes asset holdings (or wealth) at the beginning of period t , $0 < \beta < 1$ is the discount factor, $0 < \delta < 1$ is the depreciation and r is the known (and constant) interest rate. The initial assets A_0 and the initial stock of durable goods D_{-1} are given. Assume that the representative consumer has rational expectations. The instantaneous utility function is increasing in D_t and concave. The consumer derives the service flows from the stock of durable goods, so that the stock of durables, and not the consumption expenditures, is included in the utility function.

1. (5 points) Explain in words the economic interpretation of all constraints faced by the consumer.
2. (10 points) Let λ_t and μ_t denote the Lagrange multipliers for the constraints (1) and (2).
 - (a) (5 points) Write the Lagrangian for this problem. Derive the first-order necessary conditions for the consumer.

- (b) (5 points) Manipulate the first order conditions to get two equations in terms of D_t , μ_t and μ_{t+1} . (*Hint*: if your derivations are correct, you should get the “usual” Euler equation by setting $\delta = 1$). Can you describe the economic trade-off embedded in these equations?
3. (10 points) Assume that the utility function is quadratic, $U(D_t) = aC_t - bC_t^2$, and $\beta = \frac{1}{1+r}$. The range of the parameters in the utility function is such that D_t remains positive in all the periods.
- (a) (5 points) Show that the optimal conditions imply that the stock of durables D_t follows a random walk.
- (b) (5 points) Write $C_t - C_{t-1}$ in terms of $D_t - D_{t-1}$ and $D_{t-1} - D_{t-2}$. Do consumption expenditures C_t follow a random walk? What are the time series properties of C_t when $\delta = 0$?
4. (5 points) In the data, expenditures on durables appear to follow a process close to a random walk. Can you reconcile this fact with the above results?