Research Initiation Award:
The Construction of the Inscribed Polygonal Functions

Project Description
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In seeking to describe and understand some physical systems, patterns are of importance. Patterns that repeat are particularly useful because they allow predictions for future behavior. Such patterns whose values repeat at regular intervals are called periodic. Let g be a function and T be an interval of any length. The condition $\mathrm{g}(\mathrm{x} + \mathrm{T}) = \mathrm{g}(\mathrm{x})$ for every x, guarantees that g will be a periodic function. The period T of g is the smallest positive period. The solutions of many physical situations can be modeled using periodic functions. The standard periodic functions, the circular and elliptical functions, are extended to include a new class of periodic functions which will be referenced as the inscribed polygonal periodic functions. The primary goal of this research program is to formulate the construction of inscribed polygonal periodic functions. These functions are based on the geometry of equi-spaced zeros on the unit circle.

This research initiation plan is a reflection of the researcher's goal to dedicate her career to the advancement of girls and women in mathematics through faculty-mentored research. The proposed research will be centered on the construction of a new class of functions called the inscribed polygonal periodic functions. These functions, constructed from unit circle inscribed regular polygonal shapes, have not yet been explored; therefore, the research possibilities are limitless. If funded, the proposed research will be the only sponsored research program housed in the mathematics department at Clark Atlanta University. The proposed research aims to achieve the following specific objectives (1) to generalize the unit circle inscribed n-th sided regular polygon, (2) to derive properties for the square and triangular functions, (3) to recruit high school girls to study mathematics, and (4) to provide two semesters of undergraduate research to female students at Clark Atlanta University.

1. Motivation

Periodic functions values repeat at regular time intervals. While there are innumerable examples of periodic functions, two in particular are considered basic: the sine and the cosine functions (Callahan, 2008). These familiar functions are called the circular functions and are defined on a circle. A common definition follows: A circle is defined to be the collection of all points (x, y) that are equidistant from a fixed center point (h, k). The distance from the center of a circle to any point on the outer edge of the circle is called the radius r. The general equation of a circle centered at the origin with radius of length r is

$$x^2 + y^2 = r^2. \tag{1}$$

Let's suppose θ is an angle in standard position, (x, y) is a point on the terminal side of θ which lies on the unit circle, and r = 1 is the radius of the circle. Then

$$\sin\theta \equiv \frac{y}{r}, \quad \cos\theta \equiv \frac{x}{r}, \quad \operatorname{din}\theta \equiv r. \tag{2}$$

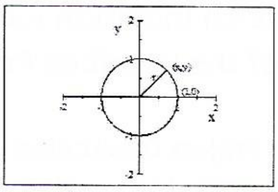

Figure 1: The sine and cosine functions defined.

Two of the three plots that arise from the geometry of a circle are pictured below.

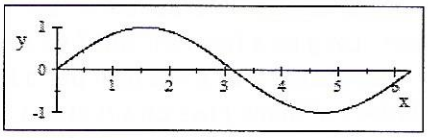

Figure 2: The plot of the sine function.

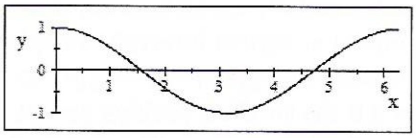

Figure 3: The plot of the cosine function.

It is worthwhile to give a brief definition of the dine function as it is not seen in a college trigonometry course. The dine function is the third function that arises from the geometry of a circle. This function evolves from the definition of the Jacobi elliptic functions. Since the circular functions are a special case of the elliptical functions we are able to define the dine function for circular functions. The dine function is defined as $\operatorname{din}\theta \equiv \frac{r}{a}$ for elliptic functions. Because $a = 1$ for circular functions, the circular dine function is constant and equal to one. Hence, a reason why the dine plot is omitted from traditional trigonometry books. However, the dine function changes for periodic functions that are not circular and is pictured in this document for the triangular functions.

1.1 Square Functions

The plots of the sine, cosine and dine functions are derived from the geometry of a circle. What happens if we inscribe a square inside of the unit circle and investigate the geometry of the square? We get functions that we call square functions. The general equation for our orientation of the square functions is $|x| + |y| \ = \pm a$, where $\pm a$ are the x and y intercepts. We normalize the equation by letting $a = 1$ as picture in figure (4).

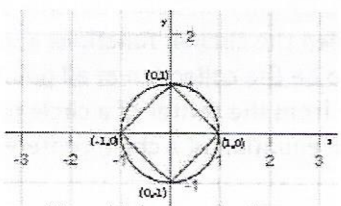

Figure 4: The square function.

Direct observations included the similarities in the graphs of the circular and square functions and treating the square functions as a dynamic system. A dynamical system is a concept in mathematics where a fixed rule describes the time dependence of a point in geometrical space. Examples include the swinging of a clock pendulum and the flow of water in a pipe (Hirsch, 2014). Dynamical systems consist of two parts, phase space and dynamics. The phase space is the collection of all possible world-states of the system in question. Each world state represents a complete snapshot of the system at some moment in time. The dynamics is a rule that transforms one point in the phase space (world state), representing the state of the system "now", into another point (world state), representing the state of the system one time unit later. In mathematical language, the dynamics is a function mapping world states into world states (Carlos, Duque 2012). The phase space for the square functions is the Cartesian plane and the dynamics are our documented results. The fundamental question that our results answer is, can the methodology for circular functions be used to generate solutions to square functions treated as a dynamic system? To answer this question we explored a class of dynamical systems called Hamiltonian systems.

These systems were first studied in mechanics around 1830 (Roussel, 2005; Tong, 2005). The way that we define the square functions lends to a direct use of the Hamiltonian system. Thus we will use the Hamiltonian's approach to determine the evolution of the square functions as a dynamic system. We now formally define a Hamiltonian system. Let a be a constant and suppose that $H(x,y) = a$ is a smooth curve with independent variables x and y in R^2. Then the equations for a Hamiltonian system are

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial x} \tag{3}$$

where H is called the Hamiltonian function or just the Hamiltonian (Tong, 2005). The specific, normalized, Hamiltonian function we use for the square functions is,

$$H(x,y) = |x| + |y| = 1 \tag{4}$$

where the dynamics of the Hamiltonian system is determined by the equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial y} = sgn(y), \quad \frac{dy}{dt} = \frac{\partial H}{\partial x} = -sgn(x) \tag{5}$$

and the sign function is defined as

$$sgn(z) = f(x) = \begin{cases} 1, & if\ z > 0, \\ 0, & if\ z = 0, \\ -1, & if\ z < 0. \end{cases} \tag{6}$$

To find the solutions to square functions written as a Hamiltonian system. We first identify a directed path in xy-phase space, $A \to B \to C \to D \to A$ as pictured in figure (5). This particular labeling of the path creates four distinct paths $A \to B,\ B \to C,\ C \to D,$ and $D \to A$. We perform calculations on each of these paths by solving the resulting equations of motion. The summarized results for the dynamic system are captured:

$$x(t+4) = x(t): \quad x(t) = f(x) = \begin{cases} 1-t, & 0 < t < 2, \\ -3+t, & 2 < t < 4, \end{cases} \tag{7}$$

$$y(t+4) = y(t): \quad y(t) = f(x) = \begin{cases} -t, & 0 < t < 1, \\ -2+t, & 1 < t < 3, \\ 1-t, & 3 < t < 4. \end{cases} \tag{8}$$

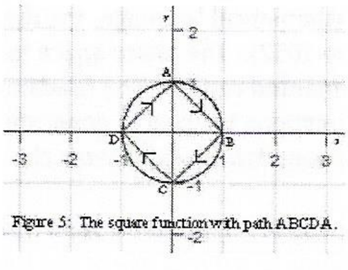

Figure 5: The square function with path ABCDA.

One cycle of the plot of these solutions is below in figures (10) and (11). An immediate observation is that the graph of the solution $x(t)$ is similar to the plot of the cosine function and the graph of the solution for $y(t)$ is similar to the plot of the minus sine function. Because of these similarities, we explored some properties of the circular functions and noticed that the square functions and the circular functions share properties.

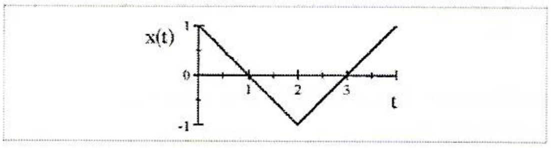

Figure 10: The plot of x(t).

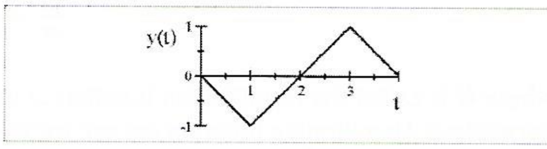

Figure 11: The plot of y(t).

1.2 Triangular Functions

What if we have three equi-spaced zeros on a triangle inscribed in a unit circle figure (8)? We get the triangular functions. Unlike the square functions the triangular functions are not symmetric about the x-axis, y-axis, and origin. The x-axis is the only axis of symmetry for the triangular functions. Thus we explore these functions using a different method.

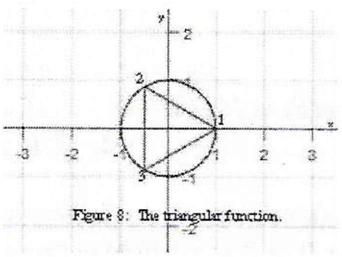

Figure 8: The triangular function.

The triangular functions are identified by the equations

$$y_{12} = -\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} \; on \; \left(-\frac{1}{2}, 0\right), \quad y_{31} = \frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}} \; on \; \left(-\frac{1}{2}, 0\right), \quad x = -\frac{1}{2} \; on \; -\frac{\sqrt{3}}{2} \leq y \leq \frac{\sqrt{3}}{2}. \tag{9}$$

The radius for each of the equations in (9) can be determine by the following set of equations:

$$r_{12} = \frac{1}{\sqrt{3}\, sin\, \theta + \cos\theta}, \quad r_{31} = \frac{1}{-\sqrt{3}\sin\theta + \cos\theta}, \quad r_{23} = -\frac{1}{2\cos\theta}. \tag{10}$$

From the sets of equations (9) and (10) we explicitly define the triangular cosine, triangular sine, and triangular dine functions on specific intervals from $0 \leq \theta < 2\pi$. The results are below.

The first interval considered is $0 \leq \theta < \frac{2\pi}{3}$. The triangular functions on this interval are

$$T_c(\theta) = \frac{\cos\theta}{\sqrt{3}\sin\theta + \cos\theta}, \quad T_s(\theta) = \frac{\sin\theta}{\sqrt{3}\sin\theta + \cos\theta}, \quad T_d(\theta) = \frac{1}{\sqrt{3}\sin\theta + \cos\theta}. \tag{11}$$

On the interval from $\frac{2\pi}{3} \leq \theta < \frac{4\pi}{3}$ we found the triangular functions to be

$$T_c(\theta) = -\frac{1}{2}, \quad T_s(\theta) = -\frac{\sin\theta}{2\cos\theta}, \quad T_d(\theta) = -\frac{1}{\cos\theta}. \tag{12}$$

The last interval to consider is the interval from $\frac{4\pi}{3} \leq \theta < 2\pi$. The triangular functions are defined as

$$T_c(\theta) = \frac{\cos\theta}{-\sqrt{3}\sin\theta + \cos\theta}, \quad T_s(\theta) = \frac{\sin\theta}{-\sqrt{3}\sin\theta + \cos\theta}, \quad T_d(\theta) = \frac{1}{-\sqrt{3}\sin\theta + \cos\theta}. \tag{13}$$

One cycle of the plot of these solutions is pictured in figures (9), (10), and (11). An immediate observation is that the plot of the solution for the triangular cosine function is similar to the graph of the cosine function and the plot of the solution for the triangular sine function is similar to the graph of the sine function.

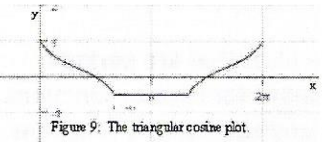
Figure 9: The triangular cosine plot.

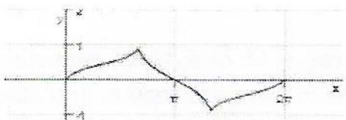
Figure 10: The triangular sine plot.

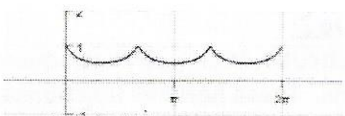
Figure11: The triangular dine function.

2. Proposed Research Activities

We were able to obtain explicit solutions for square functions and triangular functions. The methods used to explicitly define these functions were different. The square functions were treated as a dynamical system and the triangular functions were determined utilizing analytic geometry. The proposed research builds on preliminary results for the documented square functions and the triangular functions. From these research projects we formulate our research question: Can the efforts in these projects be readily generalized to the inscribed n-sided regular polygonal function? The primary goal of this project is to formulate the construction of the inscribed polygonal periodic functions.

2.1 Research Objectives for the PI

As a prelude to the discussion of the results achieved for square functions and triangular functions we introduced the circular functions which are a specific case of the elliptical functions. The circular and elliptical functions are classical and their explicit solutions have been well documented. Techniques that explicitly define the square functions are only documented by the PI and mentor, Ronald Mickens. The research team (Drs. Lewis and Mickens) has also completed a substantial amount of work defining the triangular functions. The methods used to explicitly define the square and triangular functions were different as noted in sections 1.1 and 1.2., respectfully. In fact, the method to completely define the square functions we consider somewhat simpler than the method used to define the triangular functions. A reason for the varied difficulty is that the square functions are symmetric about the x-axis, y-axis, and the origin unlike the triangular functions that are only symmetric about the x-axis.

We now summarize possible methods to achieve the specific research objectives which are (1) to generalize the unit circle inscribed n-th sided regular polygonal functions and (2) to derive properties for the square and triangular functions.

Objective 1:

Method 1: Note that the square functions and the triangular functions correspond to $n = 4$ and $n = 3$ respectively, which denote the two smallest possible polygons. At first, we plan to divide the set of n-sided polygons into two subsets (A) $n = 2k - 1$ and (B) $n = 2k$, where k is an integer greater than 2. If n falls into subset A then we will employ the method used to document the triangular functions. If n falls into subset B then we will employ the method used to define the square functions. We will continue this process for various values for n, documenting all results.

Method 2: Using the preliminary results for square and triangular functions, we will perform the method used for defining square functions on the subset where $n = 2k - 1$ and the method used to define the triangular functions on subset B, $n = 2k$, closely documenting the results.

Method 3: Determine completely new methods for explicitly defining the inscribed polygonal periodic function when $n = 5$ and $n = 6$. After documenting the results, we will use these methods on various values of n.

In a perfect world one of these methods will help to completely generalize the inscribed n-th sided regular polygonal functions. With the uncertainty of research we hypothesis that initially we will need a combination of these methods to obtain our goal of constructing the inscribed polygonal periodic functions.

Objective 2:

The circular, triangular, and square functions all have three representative plots that are periodic functions. These periodic functions are called the sine, cosine, and dine functions for circular functions. For square functions, we call these functions the square sine, square cosine, and square dine functions. For the triangular functions, we call these functions the triangular sine, triangular cosine, and triangular dine functions. The sine, square sine, and triangular sine functions all have similar plots. The cosine, square cosine, and triangular cosine functions all have similar plots. Because of the similarities, we hypothesize that the properties of the inscribed polygonal periodic functions for $n = 3$ and $n = 4$ with diminutive adjustments will follow directly from the properties of the circular functions. For the square and triangular functions, we aim to formulate the following properties with insight that follows from the circular functions. The properties are listed: (1) determine the integrals, (2) determine the derivatives, and (3) determine some of the fundamental identities (odd and even, Pythagorean, quotient, reciprocal, cofunction, sum and difference, double angle, half angle, product to sum, and sum to product identities). We also expect that more properties should arise while conducting the research.

2.2 Projected Long-Term Research Goals

The long term research goal of the PI is to build a research program that will make possible financial resources to increase the number of women and girls that choose mathematics as a career path. The proposed research, because it has not been documented, sets the stage for the PI to study additional research questions related to inscribed polygonal periodic functions. One research question that extends logically is what generalizations of the inscribed polygonal periodic functions can be made if the inscribed polygon is not regular? Another direction that excites the PI is the possibility of developing relationships between the inscribed polygonal periodic functions and other areas of mathematics. The PI plans to determine if there is a connection between the proposed research and Matroid Theory. One area that seems promising is Matroid Design. The promise comes as a result of some of the matroid designs' graphical representation being polygons. This is especially interesting because graphical representations of some matroid designs includes multiple polygonal shapes. A brief description of a matroid design follows: Young [10] reports that Murty [5] was the first to study matroids with all hyperplanes having the same size. Murty called such a matroid an "Equicardinal Matroid". Young renamed such a matroid a "Matroid Design". A hyperplane is a flat of size matroid rank minus 1. A flat is a set whose closure is equal to itself. Further work on determining properties of matroid design was completed by Edmonds, Murty, and Young [6, 11, 12]. These authors were able to connect the problem of determining the matroid designs with specified parameters with results on balanced incomplete block designs. It is a long term research goal of the PI to connect the inscribed polygonal periodic functions to matroid theory possibly through matroid designs which indicates a possible relationship that can be determined more adequately from increased literature and results on inscribed polygonal periodic functions.

2.3 Objectives for Undergraduate Students

Student participation and mentoring will be strong components of this proposal. The proposed budget includes support for six undergraduate students which consist of two students per spring and summer semesters (may change to fall and spring depending on financial need). In the fall semester, the students will be supported by LSAMP to conduct their preliminary readings. The research goal for the undergraduate students is to develop research skills that will enhance confidence in their research ability. The end result of each two student cohort project is for the student to understand how new mathematics is formulated and to accomplish something significant related to the proposed research. Students will work in pairs to replicate a collaborative effort as research should not be completed in a silo. Sample projects for undergraduate students are listed.

Project 1: A specific research question of a student cohort project, related to the proposed research follows: using the results that we have for the square functions, what differences and/or similarities are present when the square function is completely defined for an inscribed square that is rotated 45°? Determine what properties can be developed to incorporate the differences so that the rotated square will just be a special case of the square function. Complete this experiment for 30° and 60°. What generalizations can be made? The specific objectives for the research question are undergraduate student will perform various rotations of square functions in order to (1) completely define the functions, (2) document the similarities and differences of the rotated square and the original square in figure (5), (3) develop properties for the isolated cases, and (4) mentor high school students.

Project 2: Another possible research question of a student cohort project, related to proposed research follows: using the results that we have for the triangular functions, what differences and/or similarities are present when the triangular function is completely defined for an inscribed triangle that is rotated 90°? Determine what properties can be developed to incorporate the differences so that the rotated triangular will just be a special case of the triangular function. Complete this experiment for 180°, 270°, and 360°. What generalizations can be made? The specific objectives are undergraduate student cohorts will perform various rotations of each inscribed triangle in order to (1) completely define the functions, (2) document the similarities and differences of the rotated function and the original inscribed triangle in figure (9), (3) develop properties for the isolated cases, and (4) mentor high school students.

Project 3: Given the periodic wave in figure (12), by rotations and/or increasing the size of the inscribed circle and/or decreasing the size of the inscribed circle, can we find an inscribed polygonal periodic function that models the wave? The specific object for this project is to determine if an inscribed polygonal periodic function can be manipulated to match the periodic wave below. Because this wave is similar to figures (11) and (12), the student cohort will begin with manipulating the triangular functions. The undergraduate students will also mentor the high school students.