CMAT 541 Principles of Applied Math I study guide

September 2016

1

Consider

$$-u'' = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

with $f(x)$ piecewise continuous.

(a) By using Schwarz's inequality show that

$$||u||_2 \leq [\int_0^1 \int_0^1 g^2(x,t)dxdt]^{1/2}||f||_2$$

and evaluate the double integral.

(b) Show that

$$||u||_\infty \leq \frac{1}{4}||f||_1$$

2

We introduced the L_p norm

$$||f||_p = [\int_0^1 [f(x)]^p dx]^{1/p}$$

Show that if f(x) is continuous, then

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty$$

(Hint: It is easy to show that $||f||_p \le ||f||_\infty$) so that $\lim_{p\to\infty} ||f||_p \le ||f||_\infty$

Show the opposite inequality, fix $\epsilon > 0$ and note that

$$||f||_p^p = \int_0^1 |f|^p dx \ge \int_I |f|^p dx.$$

L_∞, L_1, L_2

Where I is the interval on which $|f| \ge ||f||_\infty - \epsilon$. Show then that

$$\lim_{p\to\infty} ||f||_p \ge ||f_\infty|| - \epsilon.$$

We take $\epsilon \to 0^+$ to obtain the desired result.

3

(a) Find the exact eigenvalues and λ_m and eigenfunctions $y_n(x)$ of the eigenvalue problem

$$\boxed{-u'' = \lambda u, \quad 0 < x < 1, \quad u'(0) = u(1) = 0}$$

(b) Find the eigenvalue expansion solution for

$I : [0,1] \to [-1,1]$

$$-u'' = f(x), \quad 0 < x < 1, \quad u'(0) = u(1) = 0$$

$[0,2]$

$[-\pi,\pi]$

$[0,2\pi]$

4

Consider the periodic boundary value problem given by

$$\frac{d^2u}{dx^2} + u = f(x); \quad 0 < x < 2\pi$$
$$u(0) = u(2\pi), \frac{du}{dx}(0) = \frac{du}{dx}(2\pi)$$

where $f(x)$ is a given function. Determine appropriate Green's function for this problem.

5

Let λ be an arbitrary complex number. We shall define the principal value of $\sqrt{\lambda}$ as follows. If $\sqrt{\lambda} = 0$, then $\lambda = 0$; if $\lambda \neq 0$, then λ has a unique representation $\lambda = |\lambda| e^{i\theta}$, $0 \leq \theta < 2\pi$, and the principal value of $\sqrt{\lambda}$ is defined as $\lambda^{1/2} e^{i\theta/2}$, where $|\lambda|^{1/2}$ is the positive square root of the positive real number λ. Throughout this exercise $\sqrt{\lambda}$ will stand for the principal value just defined.

(a) The general solution of $-u'' = \lambda u$ is $u(x) = A + Bx$ if $\lambda = 0$; $u(x) = Aexp(i\sqrt{\lambda x} + Bexp(-i\sqrt{\lambda x})$ if $\lambda \neq 0$ (or alternatively $u(x) = C \sin \sqrt{\lambda x} + D \cos \sqrt{\lambda x}$ for this case), show that only the real values $\lambda_n = n^2\pi^2$ are eigenvalues of this boundary value problem:

$$-u'' = \lambda u, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

(b) Find the eigenvalues and eigenfunctions of

$$-u'' = \lambda u, \quad 0 < x < 1, \quad u'(0) = u'(1) = 0.$$

(c) Obtain an eigen-function expansion solution for the problem of

$$-u'' + qu = f(x), \quad 0 < x < 1, \quad u'(0) = u'(1) = 0.$$

where q is a given positive number. What goes wrong if $q = 0$?