Of course, it is possible that this reading of the scientific evidence or this effort to estimate welfare effects is far from the mark. It is also possible that considering horizons longer than the 50 to 100 years usually examined in such studies would change the conclusions substantially. But the fact remains that most economists who have studied environmental issues seriously, even ones whose initial positions were sympathetic to environmental concerns, have concluded that the likely impact of environmental problems on growth is at most moderate.[26]

Problems

1.1. Basic properties of growth rates. Use the fact that the growth rate of a variable equals the time derivative of its log to show:

(*a*) The growth rate of the product of two variables equals the sum of their growth rates. That is, if $Z(t) = X(t)Y(t)$, then $\dot{Z}(t)/Z(t) = [\dot{X}(t)/X(t)] + [\dot{Y}(t)/Y(t)]$.

(*b*) The growth rate of the ratio of two variables equals the difference of their growth rates. That is, if $Z(t) = X(t)/Y(t)$, then $\dot{Z}(t)/Z(t) = [\dot{X}(t)/X(t)] - [\dot{Y}(t)/Y(t)]$.

(*c*) If $Z(t) = X(t)^{\alpha}$, then $\dot{Z}(t)/Z(t) = \alpha\dot{X}(t)/X(t)$.

1.2. Suppose that the growth rate of some variable, X, is constant and equal to $a > 0$ from time 0 to time t_1; drops to 0 at time t_1; rises gradually from 0 to a from time t_1 to time t_2; and is constant and equal to a after time t_2.

(*a*) Sketch a graph of the growth rate of X as a function of time.

(*b*) Sketch a graph of $\ln X$ as a function of time.

1.3. Describe how, if at all, each of the following developments affects the break-even and actual investment lines in our basic diagram for the Solow model:

(*a*) The rate of depreciation falls.

(*b*) The rate of technological progress rises.

(*c*) The production function is Cobb-Douglas, $f(k) = k^{\alpha}$, and capital's share, α, rises.

(*d*) Workers exert more effort, so that output per unit of effective labor for a given value of capital per unit of effective labor is higher than before.

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[26] This does not imply that environmental factors are always unimportant to long-run growth. Brander and Taylor (1998) make a strong case that Easter Island suffered an environmental disaster of the type envisioned by Malthusians sometime between its settlement around 400 and the arrival of Europeans in the 1700s. And they argue that other primitive societies may have also suffered such disasters.

1.4. Consider an economy with technological progress but without population growth that is on its balanced growth path. Now suppose there is a one-time jump in the number of workers.

(*a*) At the time of the jump, does output per unit of effective labor rise, fall, or stay the same? Why?

(*b*) After the initial change (if any) in output per unit of effective labor when the new workers appear, is there any further change in output per unit of effective labor? If so, does it rise or fall? Why?

(*c*) Once the economy has again reached a balanced growth path, is output per unit of effective labor higher, lower, or the same as it was before the new workers appeared? Why?

1.5. Suppose that the production function is Cobb–Douglas.

(*a*) Find expressions for k^*, y^*, and c^* as functions of the parameters of the model, s, n, δ, g, and α.

(*b*) What is the golden-rule value of k?

(*c*) What saving rate is needed to yield the golden-rule capital stock?

1.6. Consider a Solow economy that is on its balanced growth path. Assume for simplicity that there is no technological progress. Now suppose that the rate of population growth falls.

(*a*) What happens to the balanced-growth-path values of capital per worker, output per worker, and consumption per worker? Sketch the paths of these variables as the economy moves to its new balanced growth path.

(*b*) Describe the effect of the fall in population growth on the path of output (that is, total output, not output per worker).

1.7. Find the elasticity of output per unit of effective labor on the balanced growth path, y^*, with respect to the rate of population growth, n. If $\alpha_K(k^*) = \frac{1}{3}$, $g = 2\%$, and $\delta = 3\%$, by about how much does a fall in n from 2 percent to 1 percent raise y^*?

1.8. Suppose that investment as a fraction of output in the United States rises permanently from 0.15 to 0.18. Assume that capital's share is $\frac{1}{3}$.

(*a*) By about how much does output eventually rise relative to what it would have been without the rise in investment?

(*b*) By about how much does consumption rise relative to what it would have been without the rise in investment?

(*c*) What is the immediate effect of the rise in investment on consumption? About how long does it take for consumption to return to what it would have been without the rise in investment?

1.9. Factor payments in the Solow model. Assume that both labor and capital are paid their marginal products. Let w denote $\partial F(K, AL)/\partial L$ and r denote $[\partial F(K, AL)/\partial K] - \delta$.

(*a*) Show that the marginal product of labor, w, is $A[f(k) - kf'(k)]$.

(*b*) Show that if both capital and labor are paid their marginal products, constant returns to scale imply that the total amount paid to the factors of production equals total net output. That is, show that under constant returns, $wL + rK = F(K,AL) - \delta K$.

(*c*) The return to capital (r) is roughly constant over time, as are the shares of output going to capital and to labor. Does a Solow economy on a balanced growth path exhibit these properties? What are the growth rates of w and r on a balanced growth path?

(*d*) Suppose the economy begins with a level of k less than k^*. As k moves toward k^*, is w growing at a rate greater than, less than, or equal to its growth rate on the balanced growth path? What about r?

1.10. Suppose that, as in Problem 1.9, capital and labor are paid their marginal products. In addition, suppose that all capital income is saved and all labor income is consumed. Thus $\dot{K} = [\partial F(K,AL)/\partial K]K - \delta K$.

(*a*) Show that this economy converges to a balanced growth path.

(*b*) Is k on the balanced growth path greater than, less than, or equal to the golden-rule level of k? What is the intuition for this result?

1.11. Go through steps analogous to those in equations (1.28)–(1.31) to find how quickly y converges to y^* in the vicinity of the balanced growth path. (Hint: Since $y = f(k)$, we can write $k = g(y)$, where $g(\bullet) = f^{-1}(\bullet)$.)

1.12. Embodied technological progress. (This follows Solow, 1960, and Sato, 1966.) One view of technological progress is that the productivity of capital goods built at t depends on the state of technology at t and is unaffected by subsequent technological progress. This is known as *embodied technological progress* (technological progress must be "embodied" in new capital before it can raise output). This problem asks you to investigate its effects.

(*a*) As a preliminary, let us modify the basic Solow model to make technological progress capital-augmenting rather than labor-augmenting. So that a balanced growth path exists, assume that the production function is Cobb-Douglas: $Y(t) = [A(t)K(t)]^{\alpha}L(t)^{1-\alpha}$. Assume that A grows at rate μ: $\dot{A}(t) = \mu A(t)$.

Show that the economy converges to a balanced growth path, and find the growth rates of Y and K on the balanced growth path. (Hint: Show that we can write $Y/(A^{\phi}L)$ as a function of $K/(A^{\phi}L)$, where $\phi = \alpha/(1-\alpha)$. Then analyze the dynamics of $K/(A^{\phi}L)$.)

(*b*) Now consider embodied technological progress. Specifically, let the production function be $Y(t) = J(t)^{\alpha}L(t)^{1-\alpha}$, where $J(t)$ is the effective capital stock. The dynamics of $J(t)$ are given by $\dot{J}(t) = sA(t)Y(t) - \delta J(t)$. The presence of the $A(t)$ term in this expression means that the productivity of investment at t depends on the technology at t.

Show that the economy converges to a balanced growth path. What are the growth rates of Y and J on the balanced growth path? (Hint: Let $\bar{J}(t) = J(t)/A(t)$. Then use the same approach as in (*a*), focusing on $\bar{J}/(A^{\phi}L)$ instead of $K/(A^{\phi}L)$.)