Module 4

5.5. Consider the problem investigated in (5.16)–(5.21).

(*a*) Show that an increase in both w_1 and w_2 that leaves w_1/w_2 unchanged does not affect ℓ_1 or ℓ_2.

(*b*) Now assume that the household has initial wealth of amount $Z > 0$.

(*i*) Does (5.23) continue to hold? Why or why not?

(*ii*) Does the result in (*a*) continue to hold? Why or why not?

5.6. Suppose an individual lives for two periods and has utility $\ln C_1 + \ln C_2$.

(*a*) Suppose the individual has labor income of Y_1 in the first period of life and zero in the second period. Second-period consumption is thus $(1+r)(Y_1 - C_1)$; r, the rate of return, is potentially random.

(*i*) Find the first-order condition for the individual's choice of C_1.

(*ii*) Suppose r changes from being certain to being uncertain, without any change in $E[r]$. How, if at all, does C_1 respond to this change?

(*b*) Suppose the individual has labor income of zero in the first period and Y_2 in the second. Second-period consumption is thus $Y_2 - (1+r)C_1$. Y_2 is certain; again, r may be random.

(*i*) Find the first-order condition for the individual's choice of C_1.

(*ii*) Suppose r changes from being certain to being uncertain, without any change in $E[r]$. How, if at all, does C_1 respond to this change?

5.7. (*a*) Use an argument analogous to that used to derive equation (5.23) to show that household optimization requires $b/(1-\ell_t) = e^{-\rho}E_t[w_t(1+r_{t+1})b/[w_{t+1}(1-\ell_{t+1})]]$.

(*b*) Show that this condition is implied by (5.23) and (5.26). (Note that [5.26] must hold in every period.)

5.8. **A simplified real-business-cycle model with additive technology shocks.** (This follows Blanchard and Fischer, 1989, pp. 329–331.) Consider an economy consisting of a constant population of infinitely lived individuals. The representative individual maximizes the expected value of $\sum_{t=0}^{\infty} u(C_t)/(1+\rho)^t$, $\rho > 0$. The instantaneous utility function, $u(C_t)$, is $u(C_t) = C_t - \theta C_t^2$, $\theta > 0$. Assume that C is always in the range where $u'(C)$ is positive.

Output is linear in capital, plus an additive disturbance: $Y_t = AK_t + e_t$. There is no depreciation; thus $K_{t+1} = K_t + Y_t - C_t$, and the interest rate is A. Assume $A = \rho$. Finally, the disturbance follows a first-order autoregressive process: $e_t = \phi e_{t-1} + \varepsilon_t$, where $-1 < \phi < 1$ and where the ε_t's are mean-zero, i.i.d. shocks.

(*a*) Find the first-order condition (Euler equation) relating C_t and expectations of C_{t+1}.

(*b*) Guess that consumption takes the form $C_t = \alpha + \beta K_t + \gamma e_t$. Given this guess, what is K_{t+1} as a function of K_t and e_t?

(*c*) What values must the parameters α, β, and γ have for the first-order condition in part (*a*) to be satisfied for all values of K_t and e_t?

(*d*) What are the effects of a one-time shock to ε on the paths of Y, K, and C?

5.9. A simplified real-business-cycle model with taste shocks. (This follows Blanchard and Fischer, 1989, p. 361.) Consider the setup in Problem 5.8. Assume, however, that the technological disturbances (the e's) are absent and that the instantaneous utility function is $u(C_t) = C_t - \theta(C_t + \nu_t)^2$. The ν's are mean-zero, i.i.d. shocks.

(*a*) Find the first-order condition (Euler equation) relating C_t and expectations of C_{t+1}.

(*b*) Guess that consumption takes the form $C_t = \alpha + \beta K_t + \gamma \nu_t$. Given this guess, what is K_{t+1} as a function of K_t and ν_t?

(*c*) What values must the parameters α, β, and γ have for the first-order condition in (*a*) to be satisfied for all values of K_t and ν_t?

(*d*) What are the effects of a one-time shock to ν on the paths of Y, K, and C?

5.10. The balanced growth path of the model of Section 5.3. Consider the model of Section 5.3 without any shocks. Let y^*, k^*, c^*, and G^* denote the values of $Y/(AL)$, $K/(AL)$, $C/(AL)$, and $G/(AL)$ on the balanced growth path; w^* the value of w/A; ℓ^* the value of L/N; and r^* the value of r.

(*a*) Use equations (5.1)–(5.4), (5.23), and (5.26) and the fact that y^*, k^*, c^*, w^*, ℓ^*, and r^* are constant on the balanced growth path to find six equations in these six variables. (Hint: The fact that c in [5.23] is consumption per person, C/N, and c^* is the balanced-growth-path value of consumption per unit of effective labor, $C/(AL)$, implies that $c = c^*\ell^*A$ on the balanced growth path.)

(*b*) Consider the parameter values assumed in Section 5.7. What are the implied shares of consumption and investment in output on the balanced growth path? What is the implied ratio of capital to annual output on the balanced growth path?

5.11. Solving a real-business-cycle model by finding the social optimum.[34] Consider the model of Section 5.5. Assume for simplicity that $n = g = \overline{A} = \overline{N} = 0$. Let $V(K_t, A_t)$, the *value function*, be the expected present value from the current period forward of lifetime utility of the representative individual as a function of the capital stock and technology.

(*a*) Explain intuitively why $V(\bullet)$ must satisfy

$$V(K_t, A_t) = \max_{C_t, \ell_t} \{[\ln C_t + b \ln(1 - \ell_t)] + e^{-\rho} E_t[V(K_{t+1}, A_{t+1})]\}.$$

This condition is known as the *Bellman equation*.

Given the log-linear structure of the model, let us guess that $V(\bullet)$ takes the form $V(K_t, A_t) = \beta_0 + \beta_K \ln K_t + \beta_A \ln A_t$, where the values of the β's are to be determined. Substituting this conjectured form and the facts

[34] This problem uses dynamic programming and the method of undetermined coefficients. These two methods are explained in Section 10.4 and Section 5.6, respectively.

that $K_{t+1} = Y_t - C_t$ and $E_t[\ln A_{t+1}] = \rho_A \ln A_t$ into the Bellman equation yields

$$V(K_t, A_t) = \max_{C_t, \ell_t} \{[\ln C_t + b \ln(1-\ell_t)] + e^{-\rho}[\beta_0 + \beta_K \ln(Y_t - C_t) + \beta_A \rho_A \ln A_t]\}.$$

(*b*) Find the first-order condition for C_t. Show that it implies that C_t/Y_t does not depend on K_t or A_t.

(*c*) Find the first-order condition for ℓ_t. Use this condition and the result in part (*b*) to show that ℓ_t does not depend on K_t or A_t.

(*d*) Substitute the production function and the results in parts (*b*) and (*c*) for the optimal C_t and ℓ_t into the equation above for $V(\bullet)$, and show that the resulting expression has the form $V(K_t, A_t) = \beta_0' + \beta_K' \ln K_t + \beta_A' \ln A_t$.

(*e*) What must β_K and β_A be so that $\beta_K' = \beta_K$ and $\beta_A' = \beta_A$?[35]

(*f*) What are the implied values of C/Y and ℓ? Are they the same as those found in Section 5.5 for the case of $n = g = 0$?

5.12. Suppose technology follows some process other than (5.8)–(5.9). Do $s_t = \hat{s}$ and $\ell_t = \hat{\ell}$ for all t continue to solve the model of Section 5.5? Why or why not?

5.13. Consider the model of Section 5.5. Suppose, however, that the instantaneous utility function, u_t, is given by $u_t = \ln c_t + b(1-\ell_t)^{1-\gamma}/(1-\gamma)$, $b > 0, \gamma > 0$, rather than by (5.7) (see Problem 5.4).

(*a*) Find the first-order condition analogous to equation (5.26) that relates current leisure and consumption, given the wage.

(*b*) With this change in the model, is the saving rate (s) still constant?

(*c*) Is leisure per person $(1 - \ell)$ still constant?

5.14. (*a*) If the $\tilde{A}_t$'s are uniformly 0 and if $\ln Y_t$ evolves according to (5.39), what path does $\ln Y_t$ settle down to? (Hint: Note that we can rewrite [5.39] as $\ln Y_t - (n+g)t = Q + \alpha[\ln Y_{t-1} - (n+g)(t-1)] + (1-\alpha)\tilde{A}_t$, where $Q \equiv \alpha \ln \hat{s} + (1-\alpha)(\overline{A} + \ln \hat{\ell} + \overline{N}) - \alpha(n+g)$.)

(*b*) Let $\tilde{Y}_t$ denote the difference between $\ln Y_t$ and the path found in (*a*). With this definition, derive (5.40).

5.15. The derivation of the log-linearized equation of motion for capital. Consider the equation of motion for capital, $K_{t+1} = K_t + K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t - G_t - \delta K_t$.

(*a*) (*i*) Show that $\partial \ln K_{t+1}/\partial \ln K_t$ (holding $A_t, L_t, C_t,$ and G_t fixed) equals $(1 + r_{t+1})(K_t/K_{t+1})$.

(*ii*) Show that this implies that $\partial \ln K_{t+1}/\partial \ln K_t$ evaluated at the balanced growth path is $(1 + r^*)/e^{n+g}$.[36]

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[35] The calculation of β_0 is tedious and is therefore omitted.

[36] One could express r^* in terms of the discount rate ρ. Campbell (1994) argues, however, that it is easier to discuss the model's implications in terms of r^* than ρ.