$$\text{Covariance} = \text{Cov(AB)} = \sum_{i=1}^{n} (r_{Ai} - \hat{r}_A)(r_{Bi} - \hat{r}_B)P_i \tag{3-2}$$

The first term in parentheses after the Σ is the deviation of Stock A's return from its expected value under the *i*th state of the economy; the second term is Stock B's deviation under the same state; and P_i is the probability of the *i*th state occurring.

The covariance of two assets will be large and positive if their returns have large standard deviations and tend to move together; it will be large and negative for two high-σ assets that move counter to one another; and it will be small if the two assets' returns move randomly, rather than up or down with one another, or if either of the assets has a small standard deviation.

To illustrate the calculation process, first look at Table 3-1, which presents the probability distributions of the rates of return on four stocks, and at Figure 3-1, which plots scatter diagrams between returns on several pairs of the stocks. We can use Equation 3-2 to calculate the covariance between Stocks F and G as follows:

See *IFM9 Ch03 Tool Kit.xls* for all calculations.

$$\begin{aligned}
\text{Cov(FG)} &= \sum_{i=1}^{5} (r_{Fi} - \hat{r}_F)(r_{Gi} - \hat{r}_G)P_i \\
&= (6 - 10)(14 - 10)(0.1) + (8 - 10)(12 - 10)(0.2) \\
&\quad + (10 - 10)(10 - 10)(0.4) + (12 - 10)(8 - 10)(0.2) \\
&\quad + (14 - 10)(6 - 10)(0.1) \\
&= -4.8
\end{aligned}$$

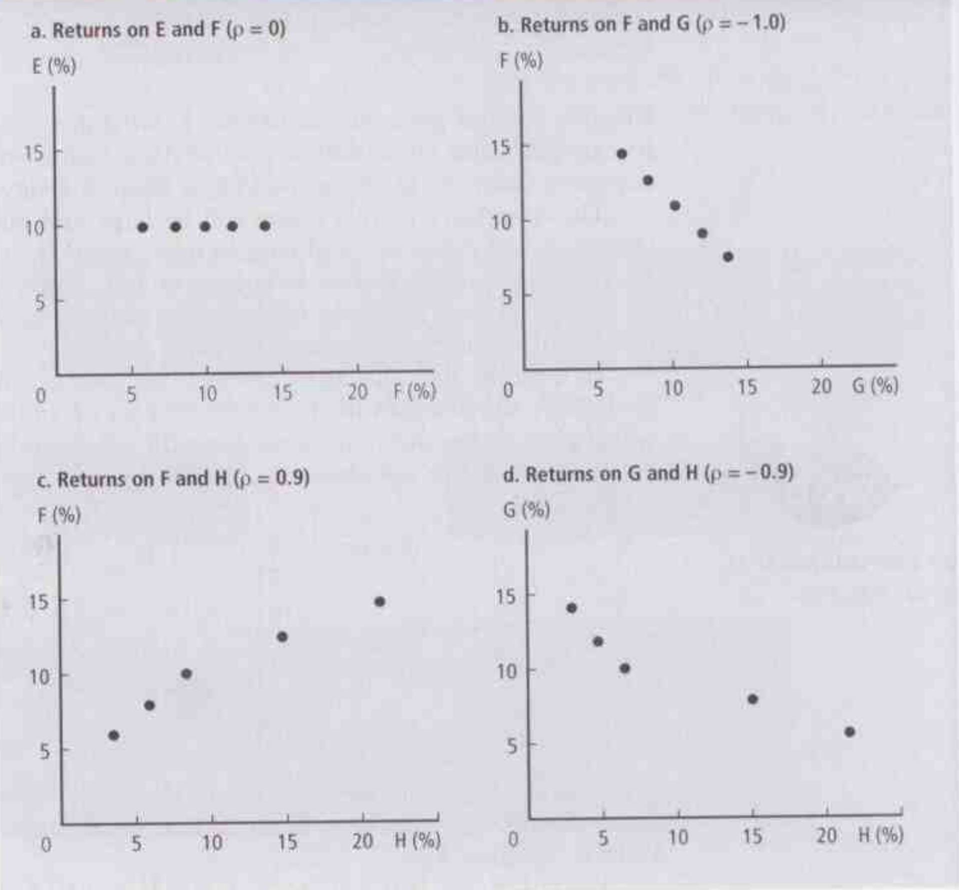
The negative sign indicates that the rates of return on Stocks F and G tend to move in opposite directions: When G's return is large, F's is small, as shown in Panel b of Figure 3-1.

The covariance between Stocks F and H is +10.8, indicating that these assets tend to move together, as shown in Panel c. A zero covariance, as between Stocks E and F, indicates that there is no relationship between the variables; that is, the

Table 3-1 *Probability Distributions of Stocks E, F, G, and H*

Probability of Occurrence		RATE OF RETURN DISTRIBUTION			
		E	F	G	H
0.1		10.0%	6.0%	14.0%	4.0%
0.2		10.0	8.0	12.0	6.0
0.4		10.0	10.0	10.0	8.0
0.2		10.0	12.0	8.0	15.0
0.1		10.0	14.0	6.0	22.0
	$\hat{r}$ =	10.0%	10.0%	10.0%	10.0%
	σ =	0.0%	2.2%	2.2%	5.3%

Figure 3-1 *Scatter Diagrams*



variables are independent. (E's return is always 10 percent, so $\sigma_E = 0\%$, and the covariance of E with any asset must be zero.)

The **correlation coefficient** standardizes the covariance, which facilitates comparisons by putting the measures on a similar scale. The correlation coefficient, ρ, is calculated as follows for variables A and B:

$$\text{Correlation coefficient (AB)} = \rho_{AB} = \frac{\text{Cov(AB)}}{\sigma_A \sigma_B} \qquad \textbf{| 3-3 |}$$

The sign of the correlation coefficient is the same as the sign of the covariance, so a positive sign means that the variables move together, a negative sign indicates that they move in opposite directions, and if ρ is close to zero, they move independently of one another. Moreover, the standardization process confines the correlation coefficient to values between -1.0 and $+1.0$. Finally, note that Equation 3-3 can be solved to find the covariance:

$$\text{Cov(AB)} = \rho_{AB}\sigma_A\sigma_B \qquad \textbf{| 3-3a |}$$

Using Equation 3-3, the correlation coefficient between Stocks F and G is −1.0 (except for rounding):

$$\rho_{FG} = \frac{-4.8}{(2.2)(2.2)} \approx -1.0$$

These two stocks are said to be perfectly negatively correlated. As Panel b of Figure 3-1 shows, we could plot a straight line on all the points. In other words, G's return is a perfect predictor of F's. Whenever the points lie exactly on a line, ρ must be equal to 1.0 if the line slopes up and equal to −1.0 if the line slopes down.

The correlation coefficient between Stocks F and H is +0.9. Thus, when H's return is large, F's return is also usually large. But H isn't a perfect predictor of F, since their plotted points do not lie exactly on a straight line.

The Two-Asset Case

If the standard deviations and correlation coefficients for the returns on the individual securities are known, a complicated-looking but operationally simple equation can be used to determine the risk of a two-asset portfolio:[2]

$$\text{Portfolio SD} = \sigma_p = \sqrt{w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2w_A(1 - w_A)\rho_{AB}\sigma_A\sigma_B} \quad | \text{ 3-4 } |$$

Here w_A is the fraction of the portfolio invested in Security A, so $(1 - w_A)$ is the fraction invested in Security B. We illustrate the equation in the next section.

Measuring Risk in Practice

Equations 3-1 and 3-2 are forward looking in the sense that they use returns in the future states of the economy and the probabilities of these future events in the calculations. However, economists and financial analysts rarely have such finely honed expectations about the future. Rather, they often use the historical standard deviations and covariances of assets as their predictions about future standard deviations and covariances. If a given portfolio's annual return over each of the previous n years has been $\bar{r}_{p1}, \bar{r}_{p2}, \ldots \bar{r}_{pn}$, then the standard deviation of these historical returns is

$$\text{Portfolio historical standard deviation} = \sigma_p = \sqrt{\frac{\sum_{i=1}^{n}(\bar{r}_{pi} - \bar{r}_{p,Avg})^2}{n - 1}} \quad | \text{ 3-1a } |$$

[2]Equation 3-4 is derived from Equation 3-1 in standard statistics books. Notice that if $w_A = 1$, all of the portfolio is invested in Security A, and Equation 3-4 reduces to σ_A:

$$\sigma_p = \sqrt{\sigma_A^2} = \sigma_A$$

The portfolio contains but a single asset, so the risk of the portfolio and that of the asset are identical. Equation 3-4 could be expanded to include any number of assets by adding additional terms, but we shall not do so here.