## 2-20 CURVED MEMBERS IN FLEXURE

The distribution of stress in a curved flexural member is determined by using the following assumptions:

- The cross section has an axis of symmetry in a plane along the length of the beam.
- Plane cross sections remain plane after bending.
- The modulus of elasticity is the same in tension as in compression.

We shall find that the neutral axis and the centroidal axis of a curved beam, unlike a straight beam, are not coincident and also that the stress does not vary linearly from the neutral axis. The notation shown in Fig. 2-27 is defined as follows:

$r_o$ = radius of outer fiber
$r_i$ = radius of inner fiber
$h$ = depth of section
$c_o$ = distance from neutral axis to outer fiber
$c_i$ = distance from neutral axis to inner fiber
$r_n$ = radius of neutral axis
$R$ = radius of centroidal axis
$e$ = distance from centroidal axis to neutral axis

Figure 2-27 shows that the neutral and centroidal axes are not coincident.* It turns out that the location of the neutral axis with respect to the center of curvature $O$ is given by the equation

$$r_n = \frac{A}{\int \frac{dA}{r}} \tag{2-64}$$

*For a complete development of the relations in this section, see Joseph E. Shigley, *Mechanical Engineering Design*, First Metric Edition, McGraw-Hill, New York, 1986, pp. 72–75.

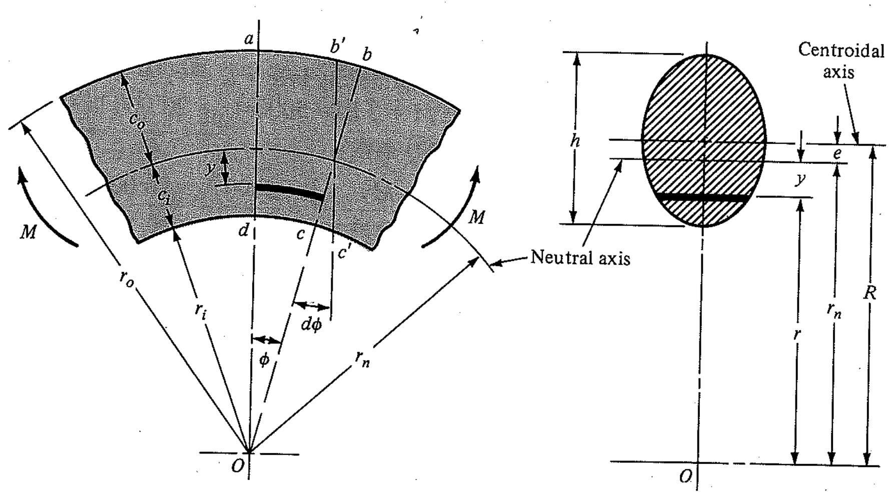


FIGURE 2-27
Note that $y$ is positive in the direction toward point $O$.

The stress distribution can be found by balancing the external applied moment against the internal resisting moment. The result is found to be

$$\sigma = \frac{My}{Ae(r_n - y)} \tag{2-65}$$

where $M$ is positive in the direction shown in Fig. 2-27. Equation (2-65) shows that the stress distribution is hyperbolic. The critical stresses occur at the inner and outer surfaces and are

$$\sigma_i = \frac{Mc_i}{Aer_i} \qquad \sigma_o = -\frac{Mc_o}{Aer_o} \tag{2-66}$$

These equations are valid for pure bending. In the usual and more general case, such as a crane hook, the U frame of a press, or the frame of a clamp, the bending moment is due to forces acting to one side of the cross section under consideration. In this case the bending moment is computed about the *centroidal axis,* not the neutral axis. Also, an additional axial tensile or compressive stress must be added to the bending stresses given by Eqs. (2-65) and (2-66) to obtain the resultant stresses acting on the section.

---

EXAMPLE 2-8 Plot the distribution of stresses across section $A$-$A$ of the crane hook shown in Fig. 2-28$a$. The cross section is rectangular, with $b = 0.75$ in and $h = 4$ in, and the load is $F = 5000$ lb.

*Solution* Since $A = bh$, we have $dA = b\,dr$ and, from Eq. (2-64),

$$r_n = \frac{A}{\int \frac{dA}{r}} = \frac{bh}{\int_{r_i}^{r_o} \frac{b}{r}\,dr} = \frac{h}{\ln \frac{r_o}{r_i}} \tag{1}$$

From Fig. 2-28$b$, we see that $r_i = 2$ in, $r_o = 6$ in, $R = 4$ in, and $A = 3\ \text{in}^2$. Thus, from Eq. (1),

$$r_n = \frac{h}{\ln (r_o/r_i)} = \frac{4}{\ln \frac{6}{2}} = 3.641 \text{ in}$$

and so the eccentricity is $e = R - r_n = 4 - 3.641 = 0.359$ in. The moment $M$ is positive and is $M = FR = 5000(4) = 20\,000$ lb · in. Adding the axial component of stress to Eq. (2-65) gives

$$\sigma = \frac{F}{A} + \frac{My}{Ae(r_n - y)} = \frac{5000}{3} + \frac{(20\,000)(3.641 - r)}{3(0.359)r} \tag{2}$$

Substituting values of $r$ from 2 to 6 in results in the stress distribution shown in Fig. 2-28$c$. The stresses at the inner and outer radii are found to be 16.9 and $-5.6$ kpsi, respectively, as shown.

---

Sections most frequently encountered in the stress analysis of curved beams are shown in Fig. 2-29. Formulas for the rectangular section were developed in Example

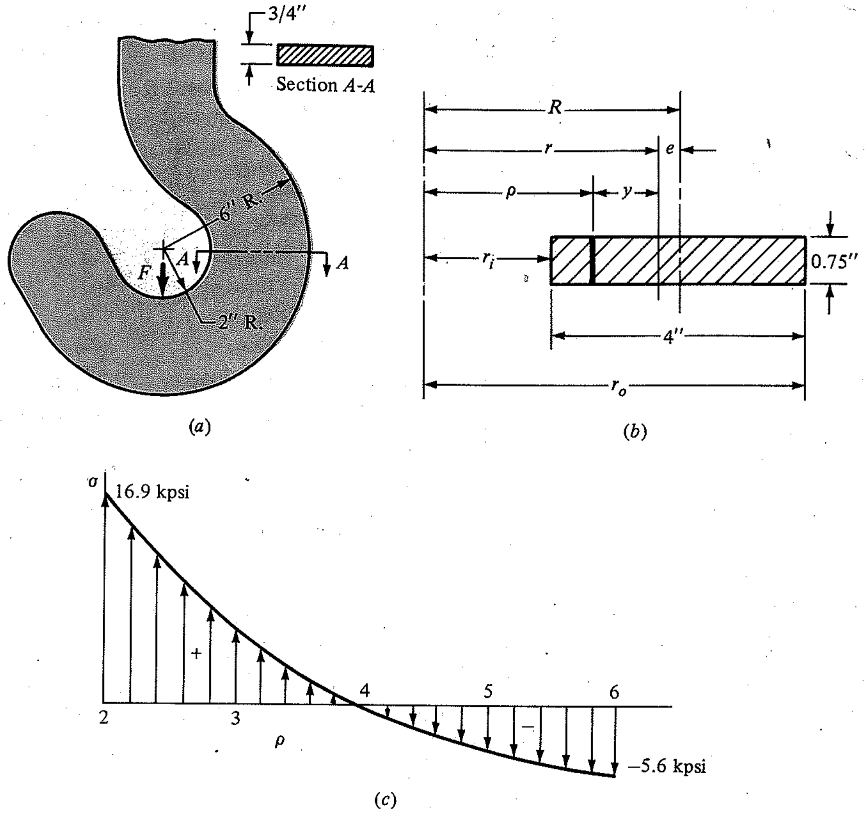


FIGURE 2-28

(*a*) Front view of crane hook; (*b*) cross section and notation; (*c*) resulting stress distribution.

2-8, but they are repeated here for convenience:

$$R = r_i + \frac{h}{2} \tag{2-67}$$

$$r_n = \frac{h}{\ln (r_o/r_i)} \tag{2-68}$$

For the trapezoidal section in Fig. 2-29*b*, the formulas are

$$R = r_i + \frac{h}{3}\,\frac{b_i + 2b_o}{b_i + b_o} \tag{2-69}$$

$$r_n = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h] \ln (r_o/r_i)} \tag{2-70}$$

For the T section in Fig. 2-29*c*, we have

$$R = r_i + \frac{b_i c_1^2 + 2b_o c_1 c_2 + b_o c_2^2}{2(b_o c_2 + b_i c_1)} \tag{2-71}$$

$$r_n = \frac{b_i c_1 + b_o c_2}{b_i \ln [(r_i + c_1)/r_i)] + b_o \ln [r_o/(r_i + c_1)]} \tag{2-72}$$

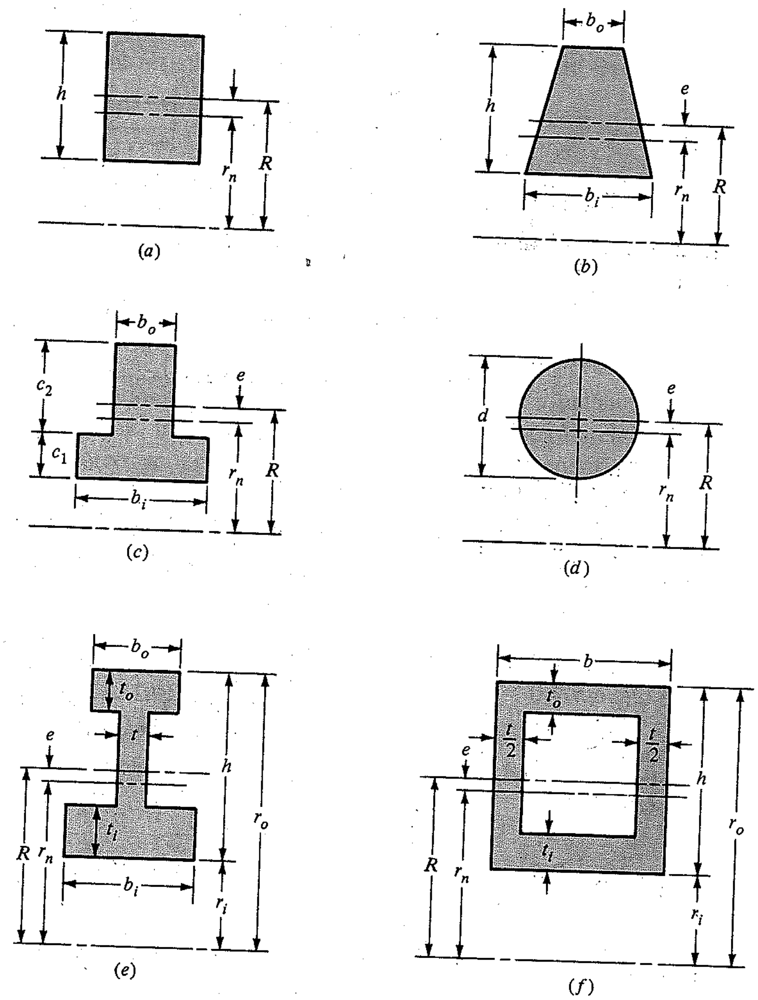


FIGURE 2-29

The equations for the solid round section of Fig. 2-29 are

$$R = r_i + \frac{d}{2} \tag{2-73}$$

$$r_n = \frac{d^2}{4(2R - \sqrt{4R^2 - d^2})} \tag{2-74}$$

For the I shape in Fig. 2-29$e$, we have

$$R = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b_i - t) + t_o(b_o - t)(h - t_o/2)}{t_i(b_i - t) + t_o(b_o - t) + ht} \tag{2-75}$$

$$r_n = \frac{t_i(b_i - t) + t_o(b_o - t) + ht_o}{b_i \ln \dfrac{r_i + t}{r_i} + t \ln \dfrac{r_o - t_o}{r_i + t_i} + b_o \ln \dfrac{r_o}{r_o - t_o}} \tag{2-76}$$

Finally, for the rectangular tubing in Fig. 2-29*f*, the results are

$$R = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b - t) + t_o(b - t)(h - t_o/2)}{ht + (b - t)(t_i + t_o)} \tag{2-77}$$

$$r_n = \frac{(b - t)(t_i + t_o) + ht}{b\left(\ln \dfrac{r_i + t_i}{r_i} + \ln \dfrac{r_o}{r_o - t_o}\right) + t \ln \dfrac{r_o - t_o}{r_i + t_i}} \tag{2-78}$$

Formulas for other sections can be obtained by performing the integration indicated by Eq. (2-64).

Many cases arise in which numerical integration must be used. These may occur because

- A digital computer is being used.
- It is not possible to integrate the function by any other means.
- The function to be integrated is described only by data.

A method of integration by Simpson's rule consists of defining equally spaced ordinates in the integration interval. Then parabolic curves are assumed to pass through each contiguous set of three ordinates. Using the notation of Fig. 2-30, the area under the curve *AB*, by Simpson's rule, is

$$I = \frac{H}{3}(Y_0 + 4Y_1 + 2Y_2 + 4Y_3 + 2Y_4 + \cdots + 4Y_{N-1} + Y_N)$$

$$= \frac{H}{3}(Y_0 + Y_N + 4\,\Sigma Y_{\text{odd}} + 2\,\Sigma Y_{\text{even}}) \tag{2-79}$$

where $H$ is the width of the interval and is

$$H = \frac{X_N - X_0}{N} \tag{2-80}$$

The terms $\Sigma Y_{\text{odd}}$ and $\Sigma Y_{\text{even}}$ are the sums, respectively, of the odd-numbered and

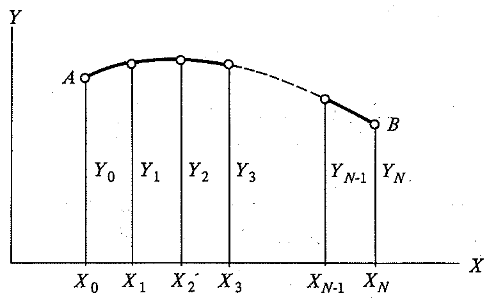


FIGURE 2-30

Notation for integration by Simpson's rule. Note that $N$ is an even number.

even-numbered subscripted ordinates. Equation (2-79) then gives Simpson's approximation to the equation

$$I = \int_{X_0}^{X_N} F(X)\, dX \tag{2-81}$$

Unfortunately, Eq. (2-79) does not always give good accuracy. The result can be checked using *Richardson's error estimate*.* This is obtained by performing the integration twice, once with all the ordinates, and again with every other ordinate. Designating the first integration by $I_1$ and the second by $I_2$, Richardson's error is

$$E = \frac{I_1 - I_2}{15} \tag{2-82}$$

The sign of the result is significant. Once $E$ has been obtained from Eq. (2-82), a better estimate of the integral is

$$I = I_1 + E \tag{2-83}$$

For the analysis of a curved beam of any arbitrary cross section, divide the cross section into an even number of strips of thickness $\Delta r$ and length $b_I$, where $b_I$ is the length of the $I$th strip. Then the equations to be solved are

$$A = \int_{r_i}^{r_o} b\, dr \tag{2-84}$$

$$R = \int_{r_i}^{r_o} \frac{br\, dr}{A} \tag{2-85}$$

$$r_n = \frac{A}{\displaystyle\int_{r_i}^{r_o} \frac{b\, dr}{r}} \tag{2-86}$$

$$e = R - r_n \tag{2-87}$$

Numerical integration methods are easy to program; see Fig. 2-31 for a simplified flow diagram.

*See B. Carnahan, H. A. Luther, and J. O Wilkes, *Applied Numerical Analysis,* Wiley, New York, 1969, p. 79.

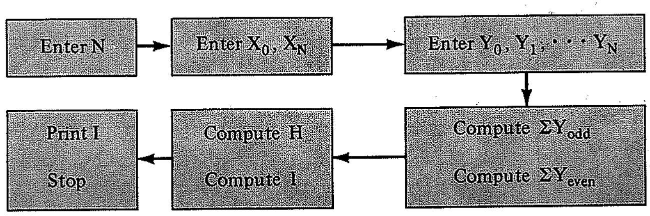


FIGURE 2-31

Flow diagram for computer solution of Simpson's rule for integration.