CHAPTER FOUR : Uncertainty Analysis : Error Propagation:

4.1 Systematic and Random Errors:

In this course, the errors associated with a given measurement are of two kinds: systematic and random. Systematic errors will simply cause a shift in the mean value but do not change the scatter. Random errors on the other hand cause the scatter around the mean.

Suppose that a thermometer reads the temperature one degree too high. The error in this case is systematic. Systematic errors can be computed through different means depending on the situation. The idea is to compare the value obtained to the value from a different source. Some devices can be calibrated, or have a way to reset the value, usually to zero (some digital scales or pressure sensors seen in the physics labs have a 'Tare' button to reset). Calibration is usually performed with a known input and the output is calibrated to what it is supposed to be. The results obtained can also be compared to results obtained from a different method. For example, one can measure the mass of an object using a scale and compare this value to the one obtained say by computing the volume and multiplying by the density. The two approaches are independent.

Yet another way of estimating the systematic error is to perform interlaboratory measurements. This simply means performing the same experient under the same conditions but at different times/places and with different personnel. Finally, experience can play a role in estimating systematic errors. With time people can gain skills that allows them to predict the results.

Calibration can help reduce the error but it can never get rid of it.

Random errors are those that manifest themselves in the scatter of data around the mean. They are also referred to as precision errors. Recall equation (2.2.1)

$$X = x \pm u \qquad (2.2.1)$$

u, called the random uncertainty, is a random error. The actual value is somewhere between $x - u$ and $x + u$. The question now is simple. What is the uncertainty that will result when this variable is used in an equation? Suppose for example that you need to figure out the volume of water in a 200 Liter barrell by emptying it with a 100 cc cup. It should be apparent that a larger error will result when compared with a 10 liter cup (with same uncertainty!). The next sections explain how to compute the total uncertainty of a measured value and the uncertainty of the final result after the mathematical operations are completed.

4.2 Design Stage Uncertainty Analysis:

The design stage uncertainty is done to get an idea on the uncertainty involved in the final result. It is performed before the experimental setup is constructed and will help in selecting the appropriate equipement.

We define the zero order uncertainty of the instrument as u_0 and is typically equal to

$$u_0 = \pm \frac{1}{2} \text{ resolution of equipement } (95\%) \quad (4.2.1)$$

The zero term implies that the variation in the results is of the order of u_0 when every other aspect of the experiment is controlled. So the scatter is expected to fall within u_0, at 95% confidence.

Often times, the manufacturer will include a systematic error u_c for the equipement. This is the uncertainty of the measuring device.

The question becomes the following: How can the two uncertainties be combined together to yield an equivalent or combined uncertainty u_x? The method used to combine these uncertainties is the root-sum-squares (RSS) method.

Suppose that a measurement of x is subject to a number k of errors, $e_1, e_2 \ldots e_k$. An estimate of the total uncertainty in the measurement, u_x is given by

$$u_x = \pm \sqrt{\sum_{k=1}^{K} e_k^2} \quad \text{(with P\% confidene)} \quad (4.2.1\text{-a})$$

$$= \pm \sqrt{e_1^2 + e_2^2 + \cdots + e_k^2} \quad \text{(with P\% confidene)} \quad (4.2.1\text{-b})$$

Usually P is about 95%. This equation assumes that the variation in an error over repeated measurements follows a normal distribution.

The student might wonder why not sum all the uncertainties together? Doing so will yield a high value for the uncertainty. In addition, this implies that all of the errors involved will be at their worst every time a measurement is made. Instead, equations (3.2.1) are the way to go. Recall $\pm u$ means somewhere betwen $-u$ & $+u$.

Since the measuring device has two sources of uncertainties, u_o and u_c, then

$$u_x = \sqrt{u_o^2 + u_c^2} \qquad (P\%) \qquad (3.2.2)$$

Example 3.2.1:

A force sensor has the following characteristics:

Resolution	0.25 N
Range	0 to 100 N
Linearity	0.20 N over range.
hysterisis	0.20 N over range.

the linearity and hysterisis values refer to the uncertainties involved in the linear relationship between measured value and output and the uncertainty associated with a repeated load.

Find the uncertainty associated with this instrument.

Solution:

4.2 Error Propagation:

To understand the concept of error propagation, consider the following example. An engineer wishes to figure out how long it will take to fill a reservoir with water using a rubber hose. She measures the flow rate of the hose by timing how long it takes to fill a certain known volume. Based on this result, the time needed to fill the reservoir is calculated. The big question becomes: how does the error associated with the flow rate measurement affect the final result (time to fill reservoir)? This is described below.

It will be assumed that x is the measured variable and $y = f(x)$ is the result needed. The sample average $\bar{X}$ (i.e. mean) and its standard deviation $S_{\bar{X}}$ can be computed as well as an approximation of the true mean μ by using equation (3.4.2)

$$\mu = \bar{X} \pm t_{\nu p} S_{\bar{X}} \qquad (3.4.2)$$

Since the result of interest is $y = f(x)$, then one must find the value $\bar{y}_1 = f(\mu)$, where μ is the real average. But since μ can only be approximated, $\bar{y}$ can only be approximated. Recall equation 3.4.2, so that

$$\bar{y} \pm u_y = f\left(\bar{X} \pm t_{\nu,p} S_{\bar{X}}\right) \qquad (4.2.1)$$

the term u_y represents the uncertainty in the y value. The objective at this point is to find the value of u_y. In order to get to the $\pm u_y$ term, consider the Taylor series expansion of $f(\bar{X} \pm t_{\nu,p} S_{\bar{X}})$

$$\bar{y} \pm u_y = f(\bar{X}) \pm \left[\left(\frac{dy}{dx}\right)_{x=\bar{X}} t_{\nu,p} S_{\bar{X}} + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_{x=\bar{X}} (tS_{\bar{X}})^2 + \cdots\right] \qquad (4.2.2)$$

This equation can be simplified further by dropping all of the second order and higher terms to get

$$\bar{y} \pm u_y = f(\bar{X}) \pm \left(\frac{dy}{dx}\right)_{x=\bar{X}} t_{\nu,p} S_{\bar{X}} \qquad (4.2.3)$$

so

$$u_y = \left(\frac{dy}{dx}\right)_{x=\bar{X}} t_{\nu,p} S_{\bar{X}} \qquad (4.2.4)$$

the notation $\frac{dY}{dx}$ represents the slope of the curve $y = f(x)$ or how much of a change in y there is for a given change in x. This is also a representation of how sensitive u_y is to changes in x and is referred to as the <u>sensitivity index</u>. This index becomes much more relevant when more than one variable are involved.

$$u_y = \left(\frac{dY}{dx}\right)_{x=\bar{x}} u_x \qquad (4.2.5)$$

Note also that, in equation (4.2.3),

$$\bar{Y} = f(\bar{X}) \qquad (4.2.6)$$

<u>Example 4.2.2</u>:

A spring has the following relationship between the force F, the stiffness k and the displacement x

$$F = kx^2$$

Find the uncertainty in F for $\Delta x = 20$ cm, $k = 50$ N/cm with $u_x = \pm 2$ cm

Solution:

In the case where the function has more than one variable, the same approach is taken. Suppose that a function T is the result needed and it depends on L number of variables

$$T = f(x_1, x_2, x_3, \ldots, x_L) \qquad (4.2.7)$$

the estimate of the true mean μ_T is given by

$$\mu_T = \bar{T} \pm u_T \quad (P\%) \qquad (4.2.8)$$

with

$$\bar{T} = f(\bar{x}_1, \bar{x}_2 ; \bar{x}_3, \ldots \bar{x}_L) \qquad (4.2.9)$$

The uncertainty in the final result can be expressed as

$$u_T = \pm \left[\Sigma (\Theta_i \, u_{\bar{X}_i})^2 \right]^{1/2} \quad (P\%) \quad (4.2.10)$$

with $\Theta_i = \left.\dfrac{\partial T}{\partial x_i}\right|_{x=\bar{X}}$ the partial derivative of T with respect to x_i at $x = \bar{X}$

$u_{\bar{X}_i}$ the uncertainty associated with the independent variable x_i

Θ_i is really nothing more than the sensitivity index for x_i

In other words, if the function has more than one variable, simply find $\Theta_i = \left.\dfrac{\partial T}{\partial x_i}\right|_{x=\bar{X}}$ for each variable, multiply it by the uncertainty for this particular x_i, square it, repeat for the remaining variables, sum the results and take the square root.

Example 4.2.3

An experimental result is $y = \beta . t$ with $u_\beta = \pm 0.01$, $u_t = \pm 0.1$. Estimate the uncertainty in y if $\beta = 5$ and $t = 10$.

Assume 95% confidence interval

Solution.

Another method exists that allows u_T to be computed. The process is as follows:

1. Find the mathematical expression for $T(x_1, x_2 \ldots x_L)$
2. Increase each variable x_i by its uncertainty, doing so one variable at a time. This will get you

$$T_1^+ = T(x_1 + u_1, x_2, x_3, \ldots, x_L) \quad (4.2.11\text{-a})$$

$$T_2^+ = T(x_1, x_2 + u_2, x_3, \ldots, x_L) \quad (4.2.11\text{-b})$$

$$\vdots$$

$$T_L^+ = T(x_1, x_2, \ldots, x_L + u_L) \quad (4.2.11\text{-c})$$

3. Repeat the same exact procedure but now subtract the uncertainty instead of adding it. This will yield

$$T_1^- = T(x_1 - u_1, x_2, x_3 \ldots x_L) \qquad (4.2.12\text{-}a)$$

$$\vdots$$

$$T_L^- = T(x_1, x_2, \ldots, x_L - u_L) \qquad (4.2.12\text{-}b)$$

4. Compute $T_0 = T(x_1, x_2, \ldots, x_L) \qquad (4.2.12\text{-}c)$

5. Compute the following

$$\delta T_i^+ = T_i^+ - T_0 \qquad (4.2.13)$$

$$\delta T_i^- = T_i^- - T_0 \qquad (4.2.14)$$

and

$$\delta T_i = \frac{\delta T_i^+ - \delta T_i^-}{2} \qquad (4.2.15)$$

the uncertainty u_T can be approximated by the previous result, or

$$u_t = \pm\left[\Sigma\left(\frac{\delta T_i^+ - \delta T_i^-}{2}\right)^2\right]^{1/2} \quad (P\%) \qquad (4.2.16)$$

<u>Example 42.4</u>:

Resolve the previous example using the results obtained above.

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Example 4.2.5:

Heat transfer from a rod of diameter D can be expressed by the Nusselt number $Nu = \frac{hD}{k}$ where h and k are the convection coefficient and the thermal conductivity, respectively. If h can be measured to within $\pm 7\%$, $D = 20 \pm 0.5$ mm and $k = 0.6 \pm 2\%$ W/m-K, find the uncertainty in Nu.

Solution:

Example 4.2.6.

Find the sensitivity index for each of the variables, then compute the uncertainty for y at $\bar{x}_1 = 5$; $\bar{x}_2 = 3$ & $\bar{x}_3 = 5$.

$$Y = 3x_1^2 - e^{x_2} + x_3$$

with $u_{x_1} = .1$; $u_{x_2} = .2$; $u_{x_3} = 0.01$.

Solution

$\Theta_1 =$

$\Theta_2 =$

$\Theta_3 =$

The uncertainty in y u_y is given by

$u_y =$

Example 4.2.7:

The shear modulus G of a given material is expressed as: $G = \dfrac{2LT}{\pi R^4 \theta}$ where L is the distance from the fixed end to the point of application of the torque, T is the torque, R is the radius and θ the angle of rotation that resulted from the applied torque T. For the following values find the uncertainty in G. Take: $L = 1\,m \pm .01\,m$ $T = 2000\,N.m \pm 2\%$ $R = 10\,cm \pm 1\%$ and $\theta = \pi/12$.

Solution: