Due Thursday, 02/11/2016 before class.

1. Itô's Formula: (5 points)
   Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. Use Itô's Formula to write the following expressions in the form $S_T^{(i)} = \text{const} + \int_0^T \ldots dt + \int_0^T \ldots dW_t$, $i = 1, 2$:

   (a) $S_T^{(0)} = (W_T)^2$.

   (b) $S_T^{(1)} = (W_T)^3$.

   (c) $S_T^{(2)} = \frac{W_T+3}{W_T^2+1}$.

   For the following problems, use the following generalized version of Itô's Formula:

   $$df(t, W_t) \quad = \quad \frac{\partial}{\partial t} f(t, W_t))\, dt + \frac{\partial}{\partial W_t} f(t, W_t)\, dW_t + \frac{1}{2} \frac{\partial^2}{\partial {W_t}^2} f(t, W_t)\, dt,$$

   so we have the first term on the right-hand side in addition to the remaining terms as in class.

2. Geometric Brownian motion: (2 + 1 points)
   Let $(W_t)_{t\geq 0}$ be a standard Brownian motion and σ, $\mu > 0$. Show that

   $$S_t = S_0 \cdot \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) \cdot t + \sigma\, W_t \right\}$$

   solves the stochastic differential equation

   $$dS_t = S_t \cdot \mu\, dt \;+\; S_t \cdot \sigma\, dW_t,\; S(0) = S_0 > 0.$$

   Under which condition is $(S_t)_{t\geq 0}$ a martingale?

3. Itô's Formula 2: (2 points)
   The stochastic process $(R_t)_{t\geq 0}$ is given by

   $$R_t = R_0\, e^{-t} + 0.05\,(1 - e^{-t}) + 0.1 \int_0^t e^{s-t} \sqrt{R_s}\, dW_s,$$

   where $(W_t)_{t\geq 0}$ is a standard Brownian motion.

   Define $X_t = R_t^2$.

   Find dX_t.