Due Thursday, 02/04/2016 before class.

1. Brownian Motion and Martingales: (1+1+1 points)

   Let $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ be two independent Brownian motions. Use the definition of Brownian motion and the definition of a martingale to show whether or not the following stochastic processes are standard Brownian motions and/or martingales, respectively (both for all three).

   (a) $\left(S_t^{(1)}\right)_{t\geq 0}$ where $S_t^{(1)} = \rho\, W_t + (1-\rho)\, Z_t$ and $0 \leq \rho \leq 1$ is a constant.

   (b) $\left(S_t^{(2)}\right)_{t\geq 0}$ where $S_t^{(2)} = W_{t+t_0} - W_{t_0}$.

   (c) $\left(S_t^{(3)}\right)_{t\geq 0}$ where $S_t^{(3)} = W_t\, Z_t$.

2. Path of Bachelier Model: (1+1 point)

   Consider the Bachelier model for the stock $(S_t)_{t\geq 0}$:

   $$S_t = S_0 + a\, t + b\, W_t,$$

   where $(W_t)_{t\geq 0}$ is a Brownian motion and $a, b > 0$.

   (a) Download daily data for the S&P 500 index for the twenty year period beginning in January 1994 until the end of 2013 (e.g., from yahoo finance). Use the data to estimate a and b for this model.

   (b) Use Excel or another spreadsheet software to simulate a (discretized) sample path of a the Bachelier model over the year 2014 using 250 equidistant time steps, and compare it to the realized path. Just hand in the resulting plot.

3. Quadratic Variation 1: (2 points)
   Let $X_t = X_0 + (\mu - 0.5\,\sigma^2)\, t + \sigma W_t$, where $(W_t)_{t\geq 0}$ is a Brownian motion. You are given the following two statements concerning X_t.

   (a) $Var[X_{t+h} - X_t] = \sigma^2\, h, t \geq 0, h \geq 0$.

   (b) $\lim_{n\to\infty} \sum_{j=1}^{n} \left[X_{\frac{jT}{n}} - X_{\frac{(j-1)T}{n}}\right]^2 = \sigma^2\, T, T \geq 0$.

   Which of them is true? Provide an explanation for your answer.

4. Quadratic Variation 2: (3 points)
   Define:

   (a) $S_t^{(1)} = [t]$, where $[t]$ is the greatest integer part of t; for example, $[3.14] = 3$, $[9.99] = 9$, and $[4] = 4$.

   (b) $S_t^{(2)} = 2t + 0.9\, W_t$, where $(W_t)_{t\geq 0}$ is a standard Brownian motion.

   (c) $S_t^{(3)} = t^2$.

   Let $h = \frac{T}{n}$ and let

   $$V_T^{(2)}(i) = \lim_{n\to\infty} \sum_{j=1}^{n} \left[S_{jh}^{(i)} - S_{(j-1)h}^{(i)}\right]^2$$

   denote the *quadratic variation* of the process $S^{(i)}$ over the time interval $[0, T]$. Rank the quadratic variations $V_T^{(2)}(1)$, $V_T^{(2)}(2)$, and $V_T^{(2)}(3)$ over the time interval $[0, 2.4]$. Provide an explanation for your answer.