two cases. If n is odd, take $k = 0$ and $m = n$. If n is even, then $n = 2b$. Since $b < n$, it is a predecessor of n, and so the inductive hypothesis allows us to assume $b = 2^\ell m$, where $\ell \ge 0$ and m is odd. The desired factorization is $n = 2b = 2^{\ell+1}m$.

Why isn't the first form of induction convenient here?

To prove uniqueness (induction is not needed here), suppose that $2^k m = n = 2^t m'$, where both k and t are nonnegative and both m and m' are odd. We may assume that $k \ge t$. If $k > t$, then canceling 2^t from both sides gives $2^{k-t}m = m'$. Since $k - t > 0$, the left side is even while the right side is odd; this contradiction shows that $k = t$. We may thus cancel 2^k from both sides, leaving $m = m'$. ■

Exercises

2.12 (i) Prove that an integer $a \ge 2$ is a perfect square if and only if whenever p is prime and $p \mid a$, then $p^2 \mid a$.

(ii) Prove that if an integer $z \ge 2$ is a perfect square and $d^4 \mid z^2$, then $d^2 \mid z$.

2.13 Let a and b be relatively prime positive integers. If ab is a perfect square, prove that both a and b are perfect squares.

2.14 * Let a, b, c, n be positive integers with $ab = c^n$. Prove that if a and b are relatively prime, then both a and b are nth powers; that is, there are positive integers k and ℓ with $a = k^n$ and $b = \ell^n$.

2.15 * For any prime p and any positive integer n, denote the highest power of p dividing n by $\mathcal{O}_p(n)$. That is,

Corollary 2.11 guarantees that $\mathcal{O}_p$ is well-defined.

$$\mathcal{O}_p(n) = e,$$

where $p^e \mid n$ but $p^{e+1} \nmid n$. If m and n are positive integers, prove that

(i) $\mathcal{O}_p(mn) = \mathcal{O}_p(m) + \mathcal{O}_p(n)$

(ii) $\mathcal{O}_p(m+n) \ge \min\{\mathcal{O}_p(m), \mathcal{O}_p(n)\}$. When does equality occur?

There is a generalization of Exercise 1.6 on page 6. Using a (tricky) inductive proof (see FCAA [26], p. 11), we can prove the ***Inequality of the Means***: if $n \ge 2$ and $a_1, \ldots, a_n$ are positive numbers, then

$$\sqrt[n]{a_1 \cdots a_n} \le \tfrac{1}{n}(a_1 + \cdots + a_n).$$

2.16 (i) Using the Inequality of the Means for $n = 3$, prove, for all triangles having a given perimeter, that the equilateral triangle has the largest area.

Hint. Use ***Heron's Formula*** for the area A of a triangle with sides of lengths a, b, c: if the ***semiperimeter*** is $s = \frac{1}{2}(a + b + c)$, then

$$A^2 = s(s-a)(s-b)(s-c).$$

(ii) What conditions on a, b, and c ensure that Heron's formula produces 0? Interpret geometrically.

2.17 Let a, b, and c be positive numbers with $a > b > c$, and let $L = \frac{1}{3}(a+b+c)$.

(i) Show that either $a > b > L > c$ or $a > L > b > c$.

(ii) Assume that $a > b > L > c$. Show that

$$L^3 - (L-b)^2 c - (L-b)(L-c)c - (L-c)^2 L = abc.$$

(iii) Use part (ii) to prove the Inequality of the Means for three variables.

(iv) Show that a box of dimensions $a \times b \times c$ can be cut up to fit inside a cube of side length L with something left over.