1. **Note:** this problem might not have "nice" constants in its solution.

Consider a mass of 1 kg attached to a spring with spring constant of 26 N/m. There is also a retarding force acting on the mass, and it is equal to 2 N per each m/sec of velocity. Additionally, an external force of $30 \cos 2t$ N acts on the mass for the first 4π seconds, and then the force vanishes. That is, the external force function is

$$f(t) = \begin{cases} 30 \cos 2t & \text{if } 0 \leq t \leq 4\pi \\ 0 & \text{if } t > 4\pi \end{cases}$$

The initial position of the mass is 0.1m and the initial velocity is 0.2 m/sec (both are in positive direction).

Set up the differential equation, and use Laplace transform methods to find the equation of the motion.

Hint 1. Express function $f(t)$ as a single-line formula with the use of unit step function(s) $u(t-a)$ first. Use graphs to make sure that you have the correct formula.

Hint 2. When you solve for the Laplace transform of the unknown function, it would look similar to $X(s) = Y_1(s) + e^{-as}Y_2(s)$, that is, there will be terms without the e^{-as}, and there will be terms with e^{-as}. You would want to do partial fractions on each of the two parts independently before you would be able to correctly find the inverse Laplace transform.

Justify your answers!

2. In this problem you will work with the homogeneous system of differential equations with constant coefficients

$$\begin{aligned} x_1'(t) &= 4x_1 - 10x_2 - 10x_3 \\ x_2'(t) &= 9x_1 - 11x_2 - 6x_3 \\ x_3'(t) &= -6x_1 + 10x_2 + 8x_3 \end{aligned}$$

which could be written as

$$\frac{dx}{dt} = Ax \quad \text{with} \quad x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad A = \begin{bmatrix} 4 & -10 & -10 \\ 9 & -11 & -6 \\ -6 & 10 & 8 \end{bmatrix}$$

(a) Find the characteristic polynomial of matrix A. Show all details of your calculations.

You want to complete this part by hand, but feel free to use technology to double-check your result.

(b) Find the eigenvalues and eigenvectors of matrix A.

Feel free to use technology to find eigenvalues, but you want to find eigenvectors by hand.

(c) Is diagonalization of matrix A possible?

If it is impossible to diagonalize the matrix, explain why, and quote the relevant theorems from the text.

If the diagonalization is possible, find the matrices P and D by hand, and then use technology to find matrix P^{-1}. Do not use decimals. Also, verify by hand that the equality $A = PDP^{-1}$ holds true – you want to evaluate the matrix product on the right hand side and see if it gives you matrix A.

(d) State the general solution to the system of differential equations

$$\frac{dx}{dt} = Ax$$

(e) Find the real form of the solution to the following initial value problem:

$$\frac{dx}{dt} = Ax, \quad x(0) = \begin{bmatrix} 4 \\ 20 \\ -2 \end{bmatrix}$$

Show all details of your calculations. You want to complete this part by hand, but feel free to use technology to double-check your results.