(1 point) In this problem you will transform this differential equation

$x'' - 5x' + 4x = 0$

into a system of linear differential equations.

You would introduce new variables

$x_1 = x, \quad x_2 = x'$

and then rewrite the equation in the matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now, take the 2×2 matrix you found above (we'll refer to it as matrix A), and find the characteristic polynomial of matrix A:

$p(\lambda) = \det(A - \lambda I) =$

Note: enter *lambda* for the variable λ

Now determine the eigenvalues of matrix A. Enter the eigenvalues as a comma-separated list.

$\lambda =$

(1 point) In this problem you will transform this differential equation
$x''' + 9x'' + 8x' = 0$
into a system of linear differential equations.

You would introduce new variables
$x_1 = x, \quad x_2 = x', \quad x_3 = x''$
and then rewrite the equation in the matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Now, take the 3×3 matrix you found above (we'll refer to it as matrix A), and find the characteristic polynomial of matrix A:
$p(\lambda) = \det(A - \lambda I) =$

Note: enter *lambda* for the variable λ

Now determine the eigenvalues of matrix A. Enter the eigenvalues as a comma-separated list.
$\lambda =$

(1 point) Consider the interaction of two species of animals in a habitat. We are told that the change of the populations $x(t)$ and $y(t)$ can be modeled by the equations

$$\frac{dx}{dt} = -11.8x + 1.8y,$$

$$\frac{dy}{dt} = -75.6x + 11.6y.$$

For this system, the smaller eigenvalue is

and the larger eigenvalue is .

The solution to the above differential equation with initial values $x(0) = 7,\ y(0) = 4$ is

$x(t) =$,

$y(t) =$.

(1 point) Consider the system of differential equations

$$\frac{dx}{dt} = -x - 0.3y,$$

$$\frac{dy}{dt} = 1.8x - 2.5y.$$

For this system, the smaller eigenvalue is

and the larger eigenvalue is .

The solution to the above differential equation with initial values $x(0) = 2,\ y(0) = 4$ is

$x(t) =$,

$y(t) =$.

(1 point) Just as there are simultaneous algebraic equations (where a pair of numbers have to satisfy a pair of equations) there are systems of differential equations, (where a pair of functions have to satisfy a pair of differential equations).
Indicate which pairs of functions satisfy this system. It will take some time to make all of the calculations.

$$y_1' = y_2 \qquad y_2' = -y_1$$

- ☐ **A.** $y_1 = e^{4x} \qquad y_2 = e^{4x}$
- ☐ **B.** $y_1 = 2e^{-2x} \qquad y_2 = 3e^{-2x}$
- ☐ **C.** $y_1 = \sin(x) + \cos(x) \qquad y_2 = \cos(x) - \sin(x)$
- ☐ **D.** $y_1 = e^{-x} \qquad y_2 = e^{-x}$
- ☐ **E.** $y_1 = \sin(x) \qquad y_2 = \cos(x)$
- ☐ **F.** $y_1 = e^{x} \qquad y_2 = e^{x}$
- ☐ **G.** $y_1 = \cos(x) \qquad y_2 = -\sin(x)$

As you can see, finding all of the solutions, particularly of a system of equations, can be complicated and time consuming. It helps greatly if we study the structure of the family of solutions to the equations. Then if we find a few solutions we will be able to predict the rest of the solutions using the structure of the family of solutions.

(1 point) Solve the system

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 6 \\ -1 & 6 \end{bmatrix} x$$

with the initial value

$$x(0) = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

(1 point) Solve the system

$$\frac{dx}{dt} = \begin{bmatrix} 20 & 10 \\ 30 & 15 \end{bmatrix} x$$

with the initial value

$$x(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

(1 point) David opens a bank account with an initial balance of 500 dollars. Let $b(t)$ be the balance in the account at time t. Thus $b(0) = 500$. The bank is paying interest at a continuous rate of 4% per year. David makes deposits into the account at a continuous rate of $s(t)$ dollars per year. Suppose that $s(0) = 500$ and that $s(t)$ is increasing at a continuous rate of 6% per year (David can save more as his income goes up over time).

(a) Set up a linear system that models the situation in the form

$$\frac{db}{dt} = m_{11}b + m_{12}s,$$

$$\frac{ds}{dt} = m_{21}b + m_{22}s.$$

$m_{11} =$,

$m_{12} =$,

$m_{21} =$,

$m_{22} =$

(b) Solve the system you set up above to find $b(t)$ and $s(t)$.

$b(t) =$,

$s(t) =$

(1 point) Solve the system

$$\frac{dx}{dt} = \begin{bmatrix} 3 & -1 \\ 10 & -3 \end{bmatrix} x, \text{ where } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

and $x(0) = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$

Give your solution in real form.

$x_1 =$,

$x_2 =$.

(1 point) Solve the system

$$\frac{dx}{dt} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix} x, \text{ where } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

and $x(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Give your solution in real form.

$x_1 =$,

$x_2 =$.

(1 point) Suppose A is a 2×2 real matrix with an eigenvalue $\lambda = 2 + 5i$ and corresponding eigenvector

$$\vec{v} = \begin{bmatrix} -1+i \\ i \end{bmatrix}.$$

Determine a fundamental set (i.e., linearly independent set) of solutions for $\vec{y}\,' = A\vec{y}$, where the fundamental set consists entirely of *real* solutions.

Enter your solutions below. Use t as the independent variable in your answers.

$$\vec{y}_1(t) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\vec{y}_2(t) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$