LEARNING OBJECTIVES

Associated Press

The storm front closed in quickly on Chicago's O'Hare Airport, shutting it down without warning. The heavy thunderstorms, lightning, and poor visibility sent American Airlines passengers and ground crew scurrying. Because American Airlines uses linear programming (LP) to schedule flights, hotels, crews, and refueling, LP has a direct impact on profitability. If American gets a major weather disruption at one of its hubs, a lot of flights may get canceled, which means a lot of crews and airplanes in the wrong places. LP is the tool that helps airlines such as American unsnarl and cope with this weather mess.

Why Use Linear Programming?

Many operations management decisions involve trying to make the most effective use of an organization's resources. Resources typically include machinery (such as planes, in the case of an airline), labor (such as pilots), money, time, and raw materials (such as jet fuel). These resources may be used to produce products (such as machines, furniture, food, or clothing) or services (such as airline schedules, advertising policies, or investment decisions). **Linear programming (LP)** is a widely used mathematical technique designed to help operations managers plan and make the decisions necessary to allocate resources.

Linear programming (LP)

Linear programming (LP)

A mathematical technique designed to help operations managers plan and make decisions necessary to allocate resources.

A few examples of problems in which LP has been successfully applied in operations management are:

- **1.** Scheduling school buses to *minimize* the total distance traveled when carrying students
- **2.** Allocating police patrol units to high crime areas to *minimize* response time to 911 calls
- **3.** Scheduling tellers at banks so that needs are met during each hour of the day while *minimizing* the total cost of labor
- **4.** Selecting the product mix in a factory to make best use of machine- and labor-hours available while *maximizing* the firm's profit
- **5.** Picking blends of raw materials in feed mills to produce finished feed combinations at *minimum* cost
- **6.** Determining the distribution system that will *minimize* total shipping cost from several warehouses to various market locations
- **7.** Developing a production schedule that will satisfy future demands for a firm's product and at the same time *minimize* total production and inventory costs
- **8.** Allocating space for a tenant mix in a new shopping mall so as to *maximize* revenues to the leasing company

Requirements of a Linear Programming Problem

All LP problems have four requirements: an objective, constraints, alternatives, and linearity:

- **1.** LP problems seek to *maximize* or *minimize* some quantity (usually profit or cost). We refer to this property as the **objective function** of an LP problem. The major objective of a typical firm is to maximize dollar profits in the long run. In the case of a trucking or airline distribution system, the objective might be to minimize shipping costs.

  ### Objective function

  A mathematical expression in linear programming that maximizes or minimizes some quantity (often profit or cost, but any goal may be used).

- **2.** The presence of restrictions, or **constraints**, limits the degree to which we can pursue our objective. For example, deciding how many units of each product in a firm's product line to manufacture is restricted by available labor and machinery. We want, therefore, to maximize or minimize a quantity (the objective function) subject to limited resources (the constraints).

  ### Constraints

  Restrictions that limit the degree to which a manager can pursue an objective.

- **3.** There must be *alternative courses of action* to choose from. For example, if a company produces three different products, management may use LP to decide how to allocate among them its limited production resources (of labor, machinery, and so on). If there were no alternatives to select from, we would not need LP.
- **4.** The objective and constraints in linear programming problems must be expressed in terms of *linear equations* or inequalities. Linearity implies proportionality and additivity. If x_1 and x_2 are decision variables, there can be no products (e.g., x_1x_2) or powers (e.g., $x_1{}^3$) in the objective or constraints.

Formulating Linear Programming Problems

STUDENT TIP

Here we set up an LP example that we will follow for most of this module.

One of the most common linear programming applications is the *product-mix problem*. Two or more products are usually produced using limited resources. The company would like to determine how many units of each product it should produce to maximize overall profit given its limited resources. Let's look at an example.

Glickman Electronics Example

The Glickman Electronics Company in Washington, DC, produces two products: (1) the Glickman x-pod, a portable music player, and (2) the Glickman BlueBerry, an internet-connected color telephone. The production process for each product is similar in that both require a certain number of hours of electronic work and a certain number of labor-hours in the assembly department. Each x-pod takes 4 hours of electronic work and 2 hours in the assembly shop. Each BlueBerry requires 3 hours in electronics and 1 hour in assembly. During the current production period, 240 hours of electronic time are available, and 100 hours of assembly department time are available. Each x-pod sold yields a profit of $7; each BlueBerry produced may be sold for a $5 profit.

Glickman's problem is to determine the best possible combination of x-pods and BlueBerrys to manufacture to reach the maximum profit. This product-mix situation can be formulated as a linear programming problem.

ACTIVE MODEL B.1

This example is further illustrated in Active Model B.1 at www.pearsonhighered.com/heizer.

We begin by summarizing the information needed to formulate and solve this problem (see Table B.1). Further, let's introduce some simple notation for use in the objective function and constraints. Let:

- X_1 = number of x-pods to be produced X_2 = number of BlueBerrys to be produced

TABLE B.1 Glickman electronics Company Problem Data

	HOURS REQUIRED TO PRODUCE ONE UNIT		
DEPARTMENT	**x-PODS (x_1)**	**BLUEBERRYS (x_2**	**AVAILABLE HOURS THIS WEEK**
Electronic	4	3	240
Assembly	2	1	100

Profit per unit	\$7	\$5

LO1 *Formulate* linear programming models, including an objective function and constraints

Now we can create the LP *objective function* in terms of X_1 and X_2:

$$\text{Maximize profit} = \$7X_1 + \$5X_2$$

Our next step is to develop mathematical relationships to describe the two constraints in this problem. One general relationship is that the amount of a resource used is to be less than or equal to (≤) the amount of resource *available*.

First constraint: Electronic time used is ≤ Electronic time available.

$$4X_1 + 3X_2 \leq 240 \text{ (hours of electronic time)}$$

Second constraint: Assembly time used is ≤ Assembly time available.

$$2X_1 + 1X_2 \leq 100 \text{ (hours of assembly time)}$$

Both these constraints represent production capacity restrictions and, of course, affect the total profit. For example, Glickman Electronics cannot produce 70 x-pods during the production period because if $X_1 = 70$, both constraints will be violated. It also cannot make $X_1 = 50$ x-pods and $X_2 = 10$ BlueBerrys. This constraint brings out another important aspect of linear programming; that is, certain interactions will exist between variables. The more units of one product that a firm produces, the fewer it can make of other products.

Graphical Solution to a Linear Programming Problem

The easiest way to solve a small LP problem such as that of the Glickman Electronics Company is the **graphical solution approach**. The graphical procedure can be used only when there are two **decision variables** (such as number of x-pods to produce, X_1, and number of BlueBerrys to produce, X_2). When there are more than two variables, it is *not* possible to plot the solution on a two-dimensional graph; we then must turn to more complex approaches described later in this module.

Graphical solution approach

A means of plotting a solution to a two-variable problem on a graph.

Decision variables

Choices available to a decision maker.

Graphical Representation of Constraints

To find the optimal solution to a linear programming problem, we must first identify a set, or region, of feasible solutions. The first step in doing so is to plot the problem's constraints on a graph.

The variable X_1 (x-pods, in our example) is usually plotted as the horizontal axis of the graph, and the variable X_2 (BlueBerrys) is plotted as the vertical axis. The complete problem may be restated as:

$$\text{Maximize profit} = \$7X_1 + \$5X_2$$

Figure B.1 Constraint A

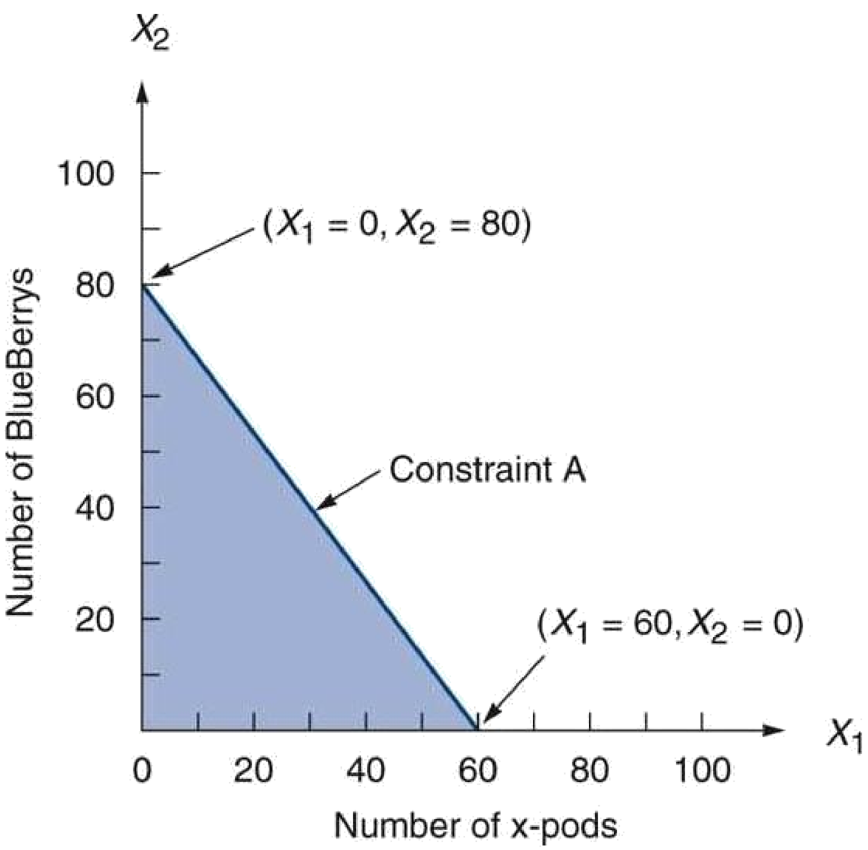
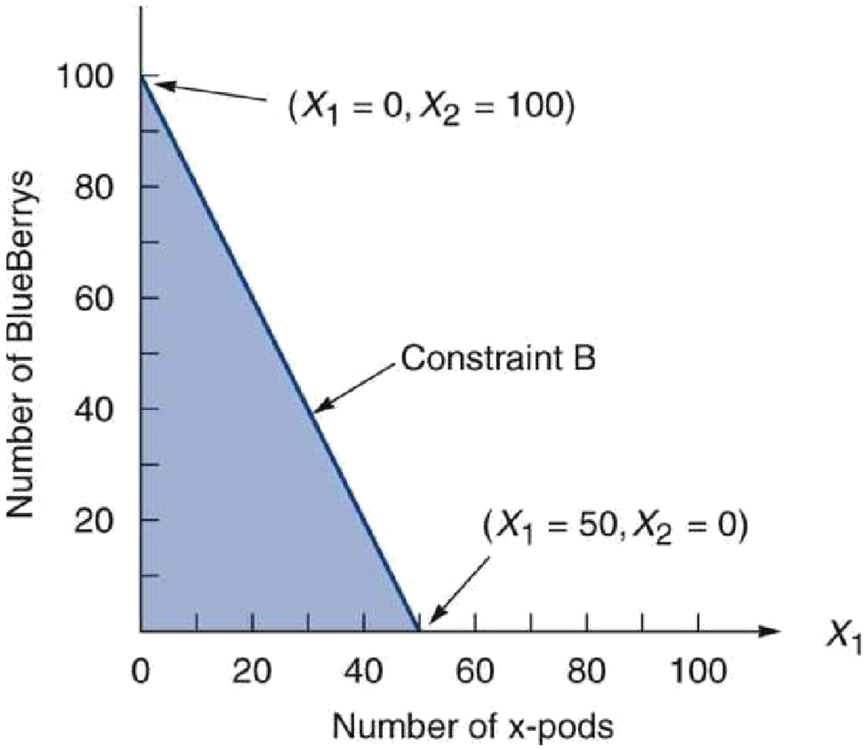


Figure B.2 Constraint B



Subject to the constraints:

- $4X_1 + 3X_2 \leq 240$ (*electronics constraint*) $2X_1 + 1X_2 \leq 100$ (*assembly constraint*)
- $X_1 \geq 0$ (*number of x-pods produced is greater than or equal to* 0) $X_2 \geq 0$ (*number of BlueBerrys produced is greater than or equal to* 0)

(These last two constraints are also called *nonnegativity constraints*.)

The first step in graphing the constraints of the problem is to convert the constraint *inequalities* into *equalities* (or equations):

- Constraint A: $4X_1 + 3X_2 = 240$ Constraint B: $2X_1 + 1X_2 = 100$

The equation for constraint A is plotted in Figure B.1 and for constraint B in Figure B.2.

To plot the line in Figure B.1, all we need to do is to find the points at which the line $4X_1 + 3X_2 = 240$ intersects the X_1 and X_2 axes. When $X_1 = 0$ (the location where the line touches the X_2 axis), it implies that $3X_2 = 240$ and that $X_2 = 80$. Likewise, when $X_2 = 0$, we see that $4X_1 = 240$ and that $X_1 = 60$. Thus, constraint A is bounded by the line running from ($X_1 = 0, X_2 = 80$) to ($X_1 = 60, X_2 = 0$). The shaded area represents all points that satisfy the original *inequality*.

Constraint B is illustrated similarly in Figure B.2. When $X_1 = 0$, then $X_2 = 100$; and when $X_2 = 0$, then $X_1 = 50$. Constraint B, then, is bounded by the line between ($X_1 = 0, X_2 = 100$) and ($X_1 = 50, X_2 = 0$). The shaded area represents the original inequality.

Figure B.3 shows both constraints together. The shaded region is the part that satisfies both restrictions. The shaded region in Figure B.3 is called the *area of feasible solutions*, or simply the **feasible region**. This region must satisfy *all* conditions specified by the program's constraints and is thus the region where all constraints overlap. Any point in the region would be a *feasible solution* to the Glickman

Electronics Company problem. Any point outside the shaded area would represent an *infeasible solution*. Hence, it would be feasible to manufacture 30 x-pods and 20 BlueBerrys ($X_1 = 30, X_2 = 20$), but it would violate the constraints to produce 70 x-pods and 40 BlueBerrys. This can be seen by plotting these points on the graph of Figure B.3.

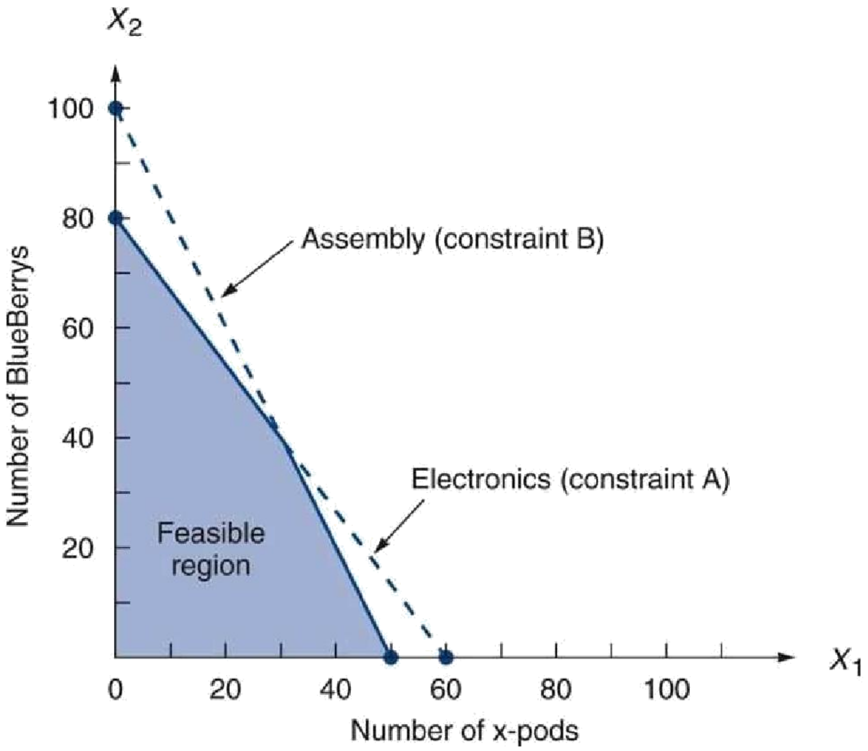
Feasible region

The set of all feasible combinations of decision variables.

Iso-Profit Line Solution Method

Now that the feasible region has been graphed, we can proceed to find the *optimal* solution to the problem. The optimal solution is the point lying in the feasible region that produces the highest profit.

Figure B.3 Feasible Solution Region for the Glickman Electronics Company Problem



Once the feasible region has been established, several approaches can be taken in solving for the optimal solution. The speediest one to apply is called the **iso-profit line method**.[1]

Iso-profit line method

An approach to solving a linear programming maximization problem graphically.

We start by letting profits equal some arbitrary but small dollar amount. For the Glickman Electronics problem, we may choose a profit of \$210. This is a profit level that can easily be obtained without violating either of the two constraints. The objective function can be written as $\$210 = 7X_1 + 5X_2$.

This expression is just the equation of a line; we call it an *iso-profit line*. It represents all combinations (of X_1, X_2) that will yield a total profit of \$210. To plot the profit line, we proceed exactly as we did to plot a constraint line. First, let $X_1 = 0$ and solve for the point at which the line crosses the X_2 axis:

$$\$210 = \$7(0) + \$5X_2$$

$$X_2 = 42 \text{ BlueBerrys}$$

LO2 *Graphically* solve an LP problem with the iso-profit line method

Then let $X_2 = 0$ and solve for X_1:

$$\$210 = \$7X_1 + \$5(0)$$

$$X_1 = 30\,x\text{ - pods}$$

We can now connect these two points with a straight line. This profit line is illustrated in Figure B.4. All points on the line represent feasible solutions that produce a profit of $210.

We see, however, that the iso-profit line for $210 does not produce the highest possible profit to the firm. In Figure B.5, we try graphing three more lines, each yielding a higher profit. The middle equation, $\$280 = \$7X_1 + \$5X_2$, was plotted in the same fashion as the lower line. When $X_1 = 0$:

$$\$280 = \$7(0) + \$5X_2$$

$$X_2 = 56 \text{ BlueBerrys}$$

When $X_2 = 0$:

$$\$280 = \$7X_1 + \$5(0)$$

$$X_1 = 40\,x\text{ - pods}$$

Again, any combination of x-pods (X_1) and BlueBerrys (X_2) on this iso-profit line will produce a total profit of $280.

[1]*Iso* means "equal" or "similar." Thus, an iso-profit line represents a line with all profits the same, in this case $210.

Figure B.4 A Profit Line of $210 Plotted for the Glickman Electronics Company

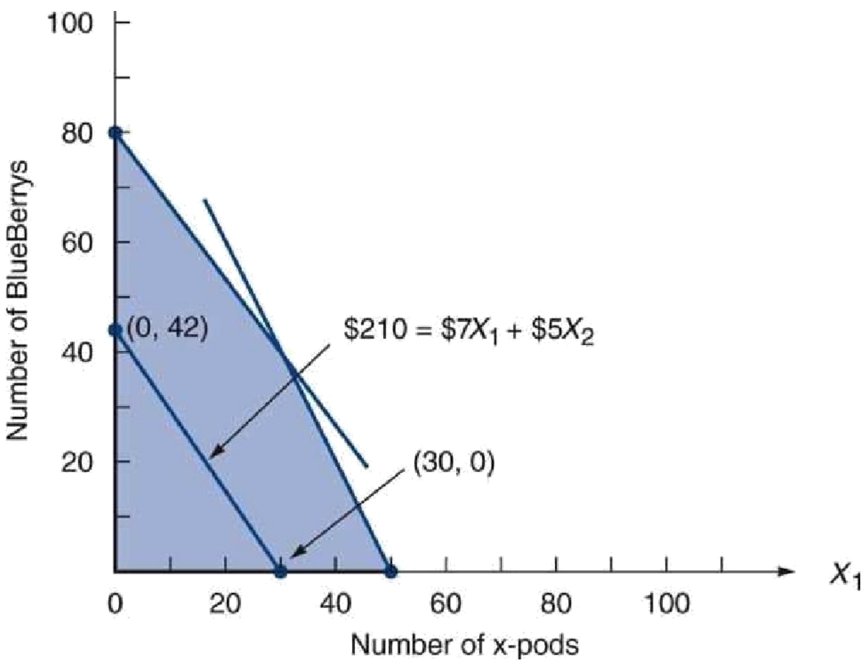
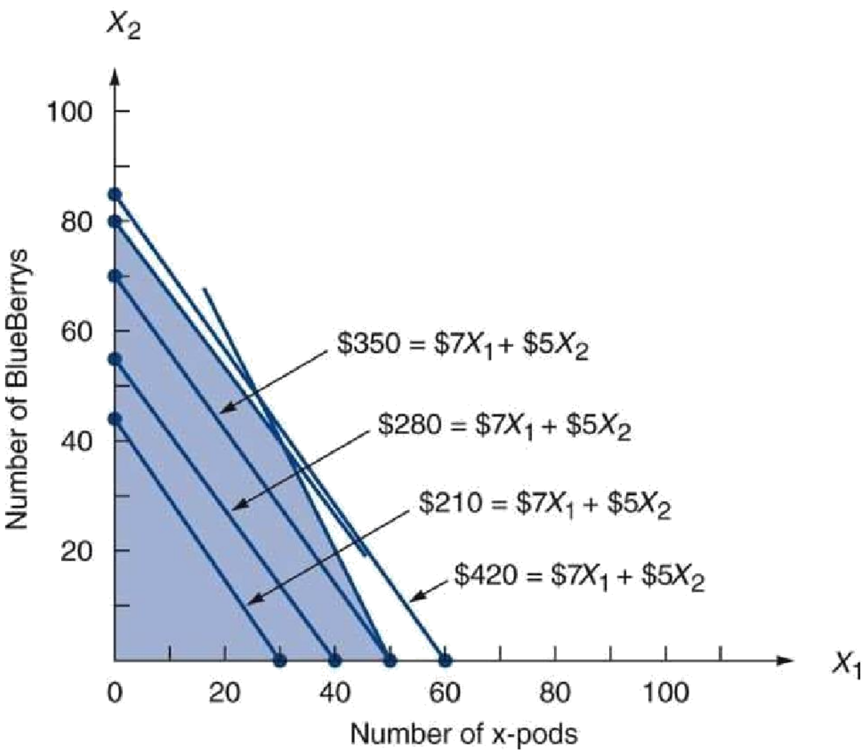


Figure B.5 Four Iso-Profit Lines Plotted for the Glickman Electronics Company



Note that the third line generates a profit of $350, even more of an improvement. The farther we move from the 0 origin, the higher our profit will be. Another important point to note is that these iso-profit lines are parallel. We now have two clues as to how to find the optimal solution to the original problem. We can draw a series of parallel profit lines (by carefully moving our ruler in a plane parallel to the first profit line). The highest profit line that still touches some point of the feasible region will pinpoint the optimal solution. Notice that the fourth line ($420) is too high to count because it does not touch the feasible region.

The highest possible iso-profit line is illustrated in Figure B.6. It touches the tip of the feasible region at the corner point ($X_1 = 30, X_2 = 40$) and yields a profit of $410.

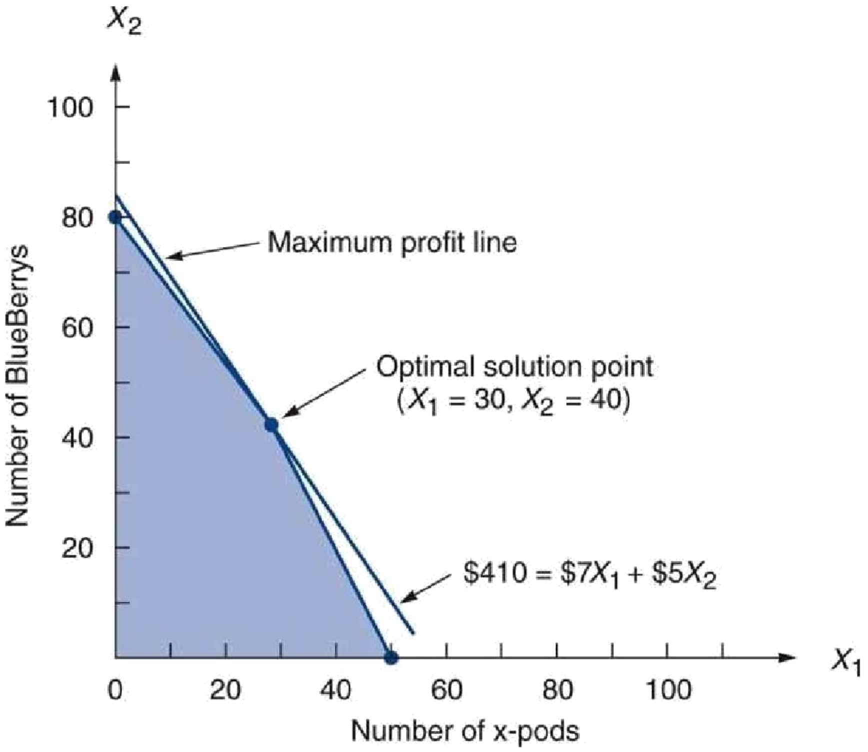
Corner-Point Solution Method

A second approach to solving linear programming problems employs the **corner-point method**. This technique is simpler in concept than the iso-profit line approach, but it involves looking at the profit at every corner point of the feasible region.

Corner-point method

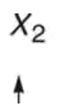
A method for solving graphical linear programming problems.

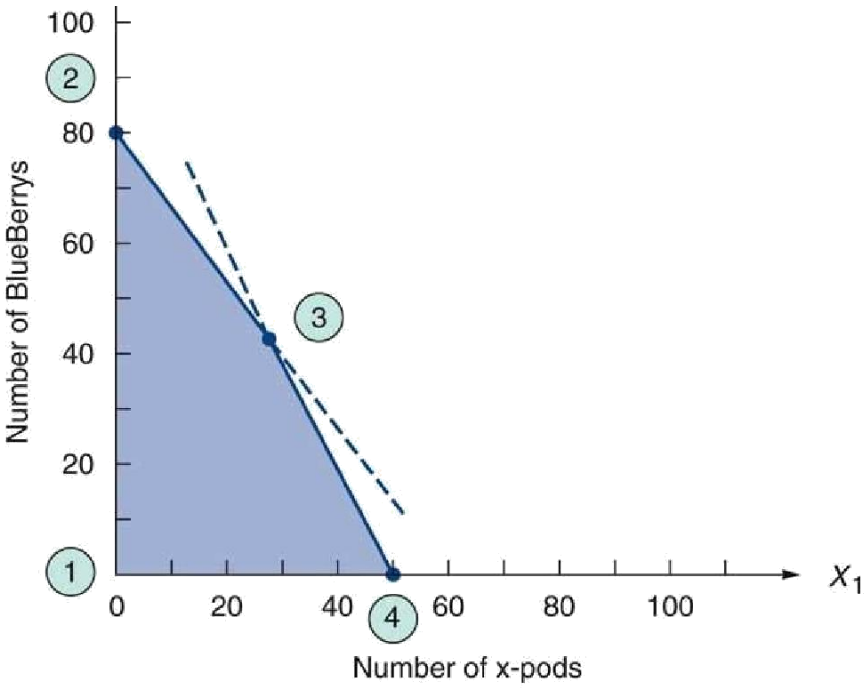
Figure B.6 Optimal Solution for the Glickman Electronics Problem



The mathematical theory behind linear programming states that an optimal solution to any problem (that is, the values of X_1, X_2 that yield the maximum profit) will lie at a *corner point*, or *extreme point*, of the feasible region. Hence, it is necessary to find only the values of the variables at each corner; the maximum profit or optimal solution will lie at one (or more) of them.

Figure B.7 The Four Corner Points of the Feasible Region





STUDENT TIP

We named the decision variables X_1 and X_2 here, but any notations (e.g., *x-p* and *B* or *X* and *Y*) would do as well.

LO3 *Graphically* solve an LP problem with the corner-point method

Once again we can see (in Figure B.7) that the feasible region for the Glickman Electronics Company problem is a four-sided polygon with four corner, or extreme, points. These points are labeled ①, ②, ③, and ④ on the graph. To find the (X_1, X_2) values producing the maximum profit, we find out what the coordinates of each corner point are, then determine and compare their profit levels:

$$\text{Point ? : } (X_1 = 0, X_2 = 0) \text{ Profit } \$7(0) + \$5(0) = \$0$$

$$\text{Point ? : } (X_1 = 0, X_2 = 80) \text{ Profit } \$7(0) + \$5(80) = \$400$$

$$\text{Point ? : } (X_1 = 50, X_2 = 0) \text{ Profit } \$7(50) + \$5(0) = \$350$$

We skipped corner point ③ momentarily because to find its coordinates *accurately*, we will have to solve for the intersection of the two constraint lines. As you may recall from algebra, we can apply the method of *simultaneous equations* to the two constraint equations:

$$4X_1 + 3X_2 = 240 \quad \text{(electronics time)}$$
$$2X_1 + 1X_2 = 100 \quad \text{(assembly time)}$$

To solve these equations simultaneously, we multiply the second equation by –2:

$$-2(2X_1 + 1X_2 = 100) = -4X_1 - 2X_2 = -200$$

$$-2(2X_1 + 1X_2 = 100) = -4X_1 - 2X_2 = -200$$

and then add it to the first equation:

$$\begin{array}{rcrcr} +4X_1 & + & 3X_2 & = & 240 \\ -4X_1 & - & 2X_2 & = & -200 \\ \hline & + & 1X_2 & = & 40 \end{array}$$

or:

$$X_2 = 40$$

Doing this has enabled us to eliminate one variable, X_1, and to solve for X_2. We can now substitute 40 for X_2 in either of the original constraint equations and solve for X_1. Let us use the first equation. When $X_2 = 40$, then:

$$\begin{aligned} 4X_1 + 3(40) &= 240 \\ 4X_1 + 120 &= 240 \\ 4X_1 &= 120 \\ X_1 &= 30 \end{aligned}$$

Thus, point [redacted] has the coordinates ($X_1 = 30$, $X_2 = 40$). We can compute its profit level to complete the analysis:

Point [redacted] : ($X_1 = 30$, $X_2 = 40$) Profit = \$7(30) + \$5(40) = \$410

Because point [redacted] produces the highest profit of any corner point, the product mix of $X_1 = 30$ x-pods and $X_2 = 40$ BlueBerrys is the optimal solution to the Glickman Electronics problem. This solution will yield a profit of \$410 per production period; it is the same solution we obtained using the iso-profit line method.

Sensitivity Analysis

Operations managers are usually interested in more than the optimal solution to an LP problem. In addition to knowing the value of each decision variable (the X_is) and the value of the objective function, they want to know how sensitive these answers are to input **parameter** changes. For example, what happens if the coefficients of the objective function are not exact, or if they change by 10% or 15%? What happens if the right-hand-side values of the constraints change? Because solutions are based on the assumption that input parameters are constant, the subject of sensitivity analysis comes into play. **Sensitivity analysis**, or postoptimality analysis, is the study of how sensitive solutions are to parameter changes.

Parameter

Numerical value that is given in a model.

Sensitivity analysis

Sensitivity analysis

An analysis that projects how much a solution may change if there are changes in the variables or input data.

There are two approaches to determining just how sensitive an optimal solution is to changes. The first is simply a trial-and-error approach. This approach usually involves resolving the entire problem, preferably by computer, each time one input data item or parameter is changed. It can take a long time to test a series of possible changes in this way.

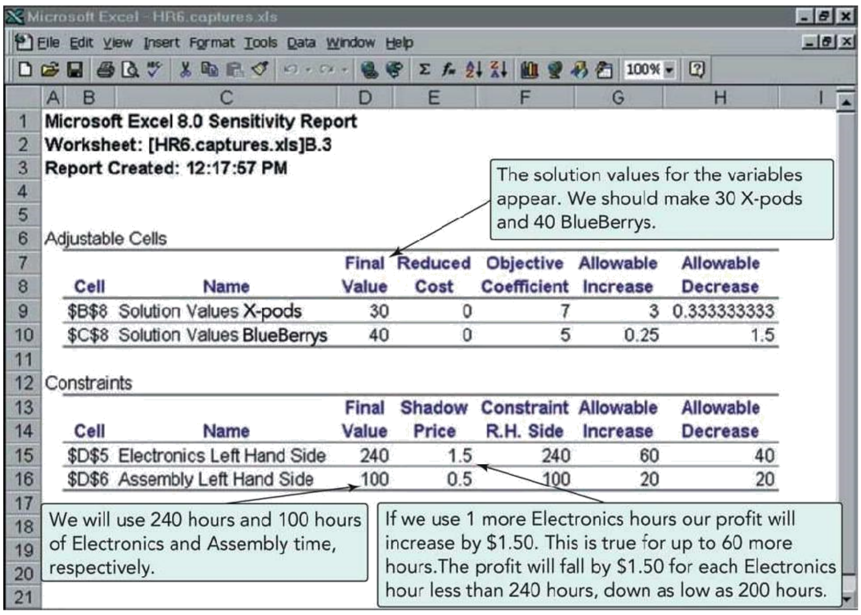
The approach we prefer is the analytic postoptimality method. After an LP problem has been solved, we determine a range of changes in problem parameters that will not affect the optimal solution or change the variables in the solution. This is done without resolving the whole problem. LP software, such as Excel's Solver or POM for Windows, has this capability. Let us examine several scenarios relating to the Glickman Electronics example.

Program B.1 is part of the Excel Solver computer-generated output available to help a decision maker know whether a solution is relatively insensitive to reasonable changes in one or more of the parameters of the problem. (The complete computer run for these data, including input and full output, is illustrated in Programs B.2 and B.3 later in this module.)

STUDENT TIP

Here we look at the sensitivity of the final answers to changing inputs.

Program B.1 Sensitivity Analysis for Glickman Electronics, Using Excel's Solver



Microsoft Excel 8.0 Sensitivity Report
Worksheet: [HR6.captures.xls]B.3
Report Created: 12:17:57 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B8	Solution Values X-pods	30	0	7	3	0.333333333
C8	Solution Values BlueBerrys	40	0	5	0.25	1.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D5	Electronics Left Hand Side	240	1.5	240	60	40
D6	Assembly Left Hand Side	100	0.5	100	20	20

Sensitivity Report

The Excel *Sensitivity Report* for the Glickman Electronics example in Program B.1 has two distinct components: (1) a table titled Adjustable Cells and (2) a table titled Constraints. These tables permit us to answer several what-if questions regarding the problem solution.

LO4 *Interpret sensitivity analysis and shadow prices*

LO4 *Interpret* sensitivity analysis and shadow prices

It is important to note that while using the information in the sensitivity report to answer what-if questions, we assume that we are considering a change to only a *single* input data value at a time. That is, the sensitivity information does not always apply to simultaneous changes in several input data values.

The *Adjustable Cells* table presents information regarding the impact of changes to the objective function coefficients (i.e., the unit profits of $7 and $5) on the optimal solution. The *Constraints* table presents information related to the impact of changes in constraint right-hand-side (RHS) values (i.e., the 240 hours and 100 hours) on the optimal solution. Although different LP software packages may format and present these tables differently, the programs all provide essentially the same information.

Changes in the Resources or Right-Hand-Side Values

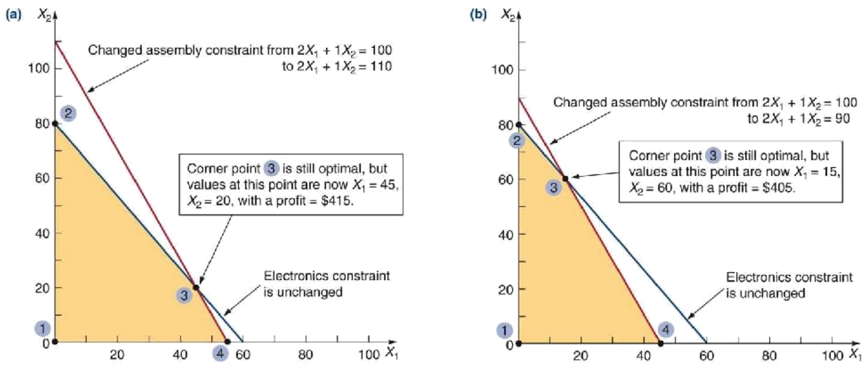
The right-hand-side values of the constraints often represent resources available to the firm. The resources could be labor-hours or machine time or perhaps money or production materials available. In the Glickman Electronics example, the two resources are hours available of electronics time and hours of assembly time. If additional hours were available, a higher total profit could be realized. How much should the company be willing to pay for additional hours? Is it profitable to have some additional electronics hours? Should we be willing to pay for more assembly time? Sensitivity analysis about these resources will help us answer these questions.

If the right-hand side of a constraint is changed, the feasible region will change (unless the constraint is redundant), and often the optimal solution will change. In the Glickman example, there were 100 hours of assembly time available each week and the maximum possible profit was $410. If the available assembly hours are *increased* to 110 hours, the new optimal solution seen in Figure B.8(a) is (45,20) and the profit is $415. Thus, the extra 10 hours of time resulted in an increase in profit of $5 or $0.50 per hour. If the hours are *decreased* to 90 hours as shown in Figure B.8(b), the new optimal solution is (15,60) and the profit is $405. Thus, reducing the hours by 10 results in a decrease in profit of $5 or $0.50 per hour. This $0.50 per hour change in profit that resulted from a change in the hours available is called the shadow price, or **dual value**. The **shadow price** for a constraint is the improvement in the objective function value that results from a one-unit increase in the right-hand side of the constraint.

Shadow price (or dual value)

The value of one additional unit of a scarce resource in LP.

Figure B.8 Glickman Electronics Sensitivity Analysis on Right-Hand-Side (RHS) Resources



Validity Range for the Shadow Price

Given that Glickman Electronics' profit increases by $0.50 for each additional hour of assembly time, does it mean that Glickman can do this indefinitely, essentially earning infinite profit? Clearly, this is illogical. How far can Glickman increase its assembly time availability and still earn an extra $0.50 profit per hour? That is, for what level of increase in the RHS value of the assembly time constraint is the shadow price of $0.50 valid?

The shadow price of $0.50 is valid as long as the available assembly time stays in a range within which all current corner points continue to exist. The information to compute the upper and lower limits of this range is given by the entries labeled Allowable Increase and Allowable Decrease in the *Sensitivity Report* in Program B.1. In Glickman's case, these values show that the shadow price of $0.50 for assembly time availability is valid for an increase of up to 20 hours from the current value and a decrease of up to 20 hours. That is, the available assembly time can range from a low of 80 (= 100 − 20) to a high of 120 (= 100 + 20) for the shadow price of $0.50 to be valid. Note that the allowable decrease implies that for each hour of assembly time that Glickman loses (up to 20 hours), its profit decreases by $0.50.

Changes in the Objective Function Coefficient

Let us now focus on the information provided in Program B.1 titled **Adjustable Cells**. Each row in the Adjustable Cells table contains information regarding a decision variable (i.e., x-pods or BlueBerrys) in the LP model.

Allowable Ranges for Objective Function Coefficients

As the unit profit contribution of either product changes, the slope of the iso-profit lines we saw earlier in Figure B.5 changes. The size of the feasible region, however, remains the same. That is, the locations of the corner points do not change.

The limits to which the profit coefficient of x-pods or BlueBerrys can be changed without affecting the optimality of the current solution is revealed by the values in the **Allowable Increase** and **Allowable Decrease** columns of the *Sensitivity Report* in Program B.1. The allowable increase in the objective function coefficient for BlueBerrys is only $0.25. In contrast, the allowable decrease is $1.50. Hence, if the unit profit of BlueBerrys drops to $4 (i.e., a decrease of $1 from the current value of $5), it is still optimal to produce 30 x-pods and 40 BlueBerrys. The total profit will drop to $370 (from $410) because each BlueBerry now yields less profit (of $1 per unit). However, if the unit profit drops below $3.50 per BlueBerry (i.e., a decrease of more than $1.50 from the current $5 profit), the current solution is no longer optimal. The LP problem will then have to be resolved using Solver, or other software, to find the new optimal corner point.

Solving Minimization Problems

Many linear programming problems involve *minimizing* an objective such as cost instead of maximizing a profit function. A restaurant, for example, may wish to develop a work schedule to meet staffing needs while minimizing the total number of employees. Also, a manufacturer may seek to distribute its products from several factories to its many regional warehouses in such a way as to minimize total shipping costs.

STUDENT TIP

LP problems can be structured to minimize costs as well as maximize profits.

Minimization problems can be solved graphically by first setting up the feasible solution region and

Minimization problems can be solved graphically by first setting up the feasible solution region and then using either the corner-point method or an iso-cost line approach (which is analogous to the iso-profit approach in maximization problems) to find the values of X_1 and X_2 that yield the minimum cost.

Example B1 shows how to solve a minimization problem.

Iso-cost

An approach to solving a linear programming minimization problem graphically.

Example B1 A MINIMIZATION PROBLEM WITH TWO VARIABLES

Cohen Chemicals, Inc., produces two types of photo-developing fluids. The first, a black-and-white picture chemical, costs Cohen $2,500 per ton to produce. The second, a color photo chemical, costs $3,000 per ton.

Based on an analysis of current inventory levels and outstanding orders, Cohen's production manager has specified that at least 30 tons of the black-and-white chemical and at least 20 tons of the color chemical must be produced during the next month. In addition, the manager notes that an existing inventory of a highly perishable raw material needed in both chemicals must be used within 30 days. To avoid wasting the expensive raw material, Cohen must produce a total of at least 60 tons of the photo chemicals in the next month.

APPROACH Formulate this information as a minimization LP problem.

Let:

- X_1 = number of tons of black-and-white photo chemical produced X_2 = number of tons of color photo chemical produced Objective: Minimize cost = $2,500$X_1$ + $3,000$X_2$

Subject to:

$$X_1 \geq 30 \text{ tons of black - and - white chemical}$$

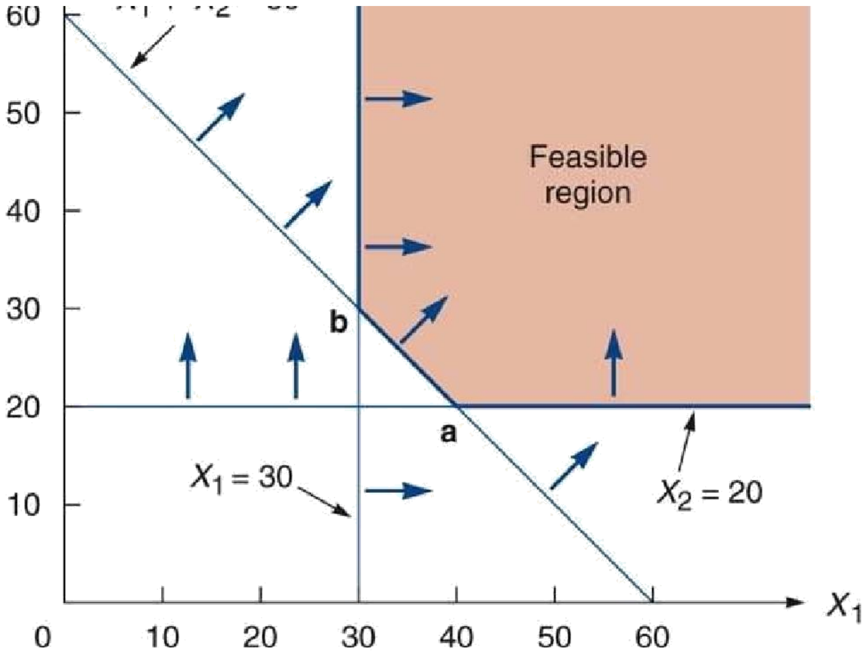
$$X_2 \geq 20 \text{ tons of color chemical}$$

$$X_1 + X_2 \geq 60 \text{ tons total}$$

$$X_1, X_2 \geq 0 \text{ nonnegativity requirements}$$

SOLUTION To solve the Cohen Chemicals problem graphically, we construct the problem's feasible region, shown in Figure B.9.

Figure B.9 Cohen Chemicals' Feasible Region



LO5 *Construct* and solve a minimization problem

Minimization problems are often unbounded outward (that is, on the right side and on the top), but this characteristic causes no problem in solving them. As long as they are bounded inward (on the left side and the bottom), we can establish corner points. The optimal solution will lie at one of the corners.

In this case, there are only two corner points, **a** and **b**, in Figure B.9. It is easy to determine that at point **a**, $X_1 = 40$ and $X_2 = 20$, and that at point **b**, $X_1 = 30$ and $X_2 = 30$. The optimal solution is found at the point yielding the lowest total cost.

Thus:

$$\begin{aligned}\text{Total cost at a} &= 2,500X_1 + 3,000X_2\\ &= 2,500(40) + 3,000(20)\\ &= \$160,000\\ \text{Total cost at b} &= 2,500X_1 + 3,000X_2\\ &= 2,500(30) + 3,000(30)\\ &= \$165,000\end{aligned}$$

The lowest cost to Cohen Chemicals is at point **a**. Hence the operations manager should produce 40 tons of the black-and-white chemical and 20 tons of the color chemical.

INSIGHT The area is either not bounded to the right or above in a minimization problem (as it is in a maximization problem).

LEARNING EXERCISE Cohen's second constraint is recomputed and should be $X_2 \geq 15$. Does anything change in the answer? [Answer: Now $X_1 = 45$, $X_2 = 15$, and total cost = \$157,500.]

RELATED PROBLEMS B.3, B.5, B.6, B.11, B.12, B.22, B.24

EXCEL OM Data File **Modbexb1.xls** can be found at www.pearsonhighered.com/heizer.

OM in Action LP at UPS

On an *average* day, the $50 billion shipping giant UPS delivers 15 million packages to 6 million customers in 220 countries. On a *really* busy day, say a few days before Christmas, it handles almost twice that number, or 300 packages per second. It does all this with a fleet of 500+ planes, making it the 11th largest commercial airline in the world.

When UPS decided it should use linear programming to map its entire operation—every pickup and delivery center and every sorting facility (more than 1,500 locations)—to find the best routes to move the millions of packages, it invested close to a decade in developing VOLCANO. This LP-based optimization system (which stands for *Volume, Location,* and *Aircraft Network Optimization*) is used to determine the least-cost set of routes, fleet assignments, and package flows.

Constraints include the number of planes, airport restrictions, and plane aircraft speed, capacity, and range.

The VOLCANO system is credited with saving UPS hundreds of millions of dollars. But that's just the start. UPS is investing $600 million more to optimize the whole supply chain to include drivers—the employees closest to the customer—so they will be able to update schedules, priorities, and time conflicts on the fly.

The UPS "airline" is not alone. Southwest runs its massive LP model (called *ILOG Optimizer)* every day to schedule its thousands of flight legs. The program has 90,000 constraints and 2 million variables. Continental's LP program is called *OptSolver,* and Delta's is called *Coldstart.* Airlines, like many other firms, manage their millions of daily decisions with LP.

Sources: Aviation Daily (February 9, 2004); **Compass.ups.com** (May 2008); and *Interfaces* (January–February 2004).

Linear Programming Applications

STUDENT TIP

Now we look at three larger problems—ones that have more than two decision variables each and therefore are not graphed.

The foregoing examples each contained just two variables (X_1 and X_2). Most real-world problems (as we see in the UPS *OM in Action* box above) contain many more variables, however. Let's use the principles already developed to formulate a few more-complex problems. The practice you will get by "paraphrasing" the following LP situations should help develop your skills for applying linear programming to other common operations situations.

Production-Mix Example

Example B2 involves another *production-mix* decision. Limited resources must be allocated among various products that a firm produces. The firm's overall objective is to manufacture the selected products in such quantities as to maximize total profits.

LO6 *Formulate* production-mix, diet, and labor scheduling problems

Example B2 A PRODUCTION-MIX PROBLEM

Failsafe Electronics Corporation primarily manufactures four highly technical products, which it supplies to aerospace firms that hold NASA contracts. Each of the products must pass through the following departments before they are shipped: wiring, drilling, assembly, and inspection. The time requirements in each department (in hours) for each unit produced and its corresponding profit value are summarized in this table:

	DEPARTMENT				
PRODUCT	**WIRING**	**DRILLING**	**ASSEMBLY**	**INSPECTION**	**UNIT PROFIT**
XJ201	.5	3	2	.5	$ 9
XM897	1.5	1	4	1.0	$12
TR29	1.5	2	1	.5	$15
BR788	1.0	3	2	.5	$11

The production time available in each department each month and the minimum monthly production requirement to fulfill contracts are as follows:

DEPARTMENT	**CAPACITY (HOURS)**	**PRODUCT**	**MINIMUM PRODUCTION LEVEL**
Wiring	1,500	XJ201	150
Drilling	2,350	XM897	100
Assembly	2,600	TR29	200
Inspection	1,200	BR788	400

APPROACH Formulate this production-mix situation as an LP problem. The production manager first specifies production levels for each product for the coming month. He lets:

- X_1 = number of units of XJ201 produced X_2 = number of units of XM897 produced X_3 = number of units of TR29 produced X_4 = number of units of BR788 produced

SOLUTION The LP formulation is:

Objective: Maximize profit = $9X_1 + 12X_2 + 15X_3 + 11X_4$

subject to:

$$.5X_1 + 1.5X_2 + 1.5X_3 + 1X_4 \leq 15,00 \text{ hours of wiring available}$$

$$3X_1 + 1X_2 + 2X_3 + 3X_4 \leq 2,350 \text{ hours of drilling availabel}$$

$$2X_1 + 4X_2 + 1X_3 + 2X_4 \leq 2,600 \text{ hours of assembly availabel}$$

$$.5X_1 + 1X_2 + .5X_3 + .5X_4 \leq 1,200 \text{ hours of inspection}$$

$$X_1 \geq 150 \text{ unit of XJ 201}$$

$$X_2 \geq 100 \text{ units of XM 897}$$

$$X_3 \geq 200 \text{ units of TR 29}$$

$$X_4 \geq 400 \text{ units of BR 788}$$

$$X_1, X_2, X_3, X_4 \geq 0$$

INSIGHT There can be numerous constraints in an LP problem. The constraint right-hand sides may be in different units, but the objective function uses one common unit—dollars of profit, in this case. Because there are more than two decision variables, this problem is not solved graphically.

LEARNING EXERCISE Solve this LP problem as formulated. What is the solution? [Answer: X_1 = 150, X_2 = 300, X_3 = 200, X_4 = 400.]

RELATED PROBLEMS B.7, B.8, B.10, B.19, B.20, B.21, B.23, B.28

Diet Problem Example

Example B3 illustrates the *diet problem*, which was originally used by hospitals to determine the most economical diet for patients. Known in agricultural applications as the *feed-mix problem*, the diet problem involves specifying a food or feed ingredient combination that will satisfy stated nutritional requirements at a minimum cost level.

Example B3 A DIET PROBLEM

The Feed 'N Ship feedlot fattens cattle for local farmers and ships them to meat markets in Kansas City and Omaha. The owners of the feedlot seek to determine the amounts of cattle feed to buy to satisfy minimum nutritional standards and, at the same time, minimize total feed costs.

Each grain stock contains different amounts of four nutritional ingredients: A, B, C, and D. Here are the ingredient contents of each grain, in *ounces per pound of grain*:

	FEED		
INGREDIENT	STOCK X	STOCK Y	STOCK Z

A	3 oz	2 oz	4 oz
B	2 oz	3 oz	1 oz
C	1 oz	0 oz	2 oz
D	6 oz	8 oz	4 oz

The cost per pound of grains X, Y, and Z is $0.02, $0.04, and $0.025, respectively. The minimum requirement per cow per month is 64 ounces of ingredient A, 80 ounces of ingredient B, 16 ounces of ingredient C, and 128 ounces of ingredient D.

The feedlot faces one additional restriction—it can obtain only 500 pounds of stock Z per month from the feed supplier, regardless of its need. Because there are usually 100 cows at the Feed ‘N Ship feedlot at any given time, this constraint limits the amount of stock Z for use in the feed of each cow to no more than 5 pounds, or 80 ounces, per month.

APPROACH Formulate this as a minimization LP problem.

Let: X_1 = number of pounds of stock X purchased per cow each month

X_2 = number of pounds of stock Y purchased per cow each month

X_3 = number of pounds of stock Z purchased per cow each month

SOLUTION

Objective: Minimize cost = $.02X_1 + .04X_2 + .025X_3$

subject to:

Ingredient A requirement: $3X_1 + 2X_2 + 4X_3 \geq 64$

Ingredient B requirement: $2X_1 + 3X_2 + 1X_3 \geq 80$

Ingredient C requirement: $1X_1 + 0X_2 + 2X_3 \geq 16$

Ingredient D requirement: $6X_1 + 8X_2 + 4X_3 \geq 128$

Stock Z limitation: $X_3 \leq 5$

Stock Z limitation: $X_3 \le$ [illegible]

$$X_1, X_2, X_3 \ge 0$$

The cheapest solution is to purchase 40 pounds of grain X_1, at a cost of $0.80 per cow.

INSIGHT Because the cost per pound of stock X is so low, the optimal solution excludes grains Y and Z.

LEARNING EXERCISE The cost of a pound of stock X just increased by 50%. Does this affect the solution? [Answer: Yes, when the cost per pound of grain X is $0.03, X_1 = 16 pounds, X_2 = 16 pounds, X_3 = 0, and cost = $1.12 per cow.]

RELATED PROBLEMS B.6, B.30

Labor Scheduling Example

Labor scheduling problems address staffing needs over a specific time period. They are especially useful when managers have some flexibility in assigning workers to jobs that require overlapping or interchangeable talents. Large banks and hospitals frequently use LP to tackle their labor scheduling. Example B4 describes how one bank uses LP to schedule tellers.

Example B4 SCHEDULING BANK TELLERS

Mexico City Bank of Commerce and Industry is a busy bank that has requirements for between 10 and 18 tellers depending on the time of day. Lunchtime, from noon to 2 p.m., is usually heaviest. The table below indicates the workers needed at various hours that the bank is open.

TIME PERIOD	NUMBER OF TELLERS REQUIRED	TIME PERIOD	NUMBER OF TELLERS REQUIRED
9 a.m.–10 a.m.	10	1 p.m.–2 p.m.	18
10 a.m.–11 a.m.	12	2 p.m.–3 p.m.	17
11 a.m.–Noon	14	3 p.m.–4 p.m.	15
Noon–1 p.m.	16	4 p.m.–5 p.m.	10

The bank now employs 12 full-time tellers, but many people are on its roster of available part-time employees. A part-time employee must put in exactly 4 hours per day but can start anytime between 9 a.m. and 1 p.m. Part-timers are a fairly inexpensive labor pool because no retirement or lunch benefits are provided them. Full-timers, on the other hand, work from 9 a.m. to 5 p.m. but are allowed 1 hour for lunch. (Half the full-timers eat at 11 a.m., the other half at noon.) Full-timers thus provide 35 hours per week of productive labor time.

By corporate policy, the bank limits part-time hours to a maximum of 50% of the day's total requirement. Part-timers earn $6 per hour (or $24 per day) on average, whereas full-timers earn $75 per day in salary and benefits on average.

APPROACH The bank would like to set a schedule, using LP, that would minimize its total manpower costs. It will release 1 or more of its full-time tellers if it is profitable to do so.

We can let:

F = full-time tellers

P_1 = part-timers starting at 9 a.m. (leaving at 1 p.m.)

P_2 = part-timers starting at 10 a.m. (leaving at 2 p.m.)

P_3 = part-timers starting at 11 a.m. (leaving at 3 p.m.)

P_4 = part-timers starting at noon (leaving at 4 p.m.)

P_5 = part-timers starting at 1 p.m. (leaving at 5 p.m.)

SOLUTION Objective function:

Minimize total daily manpower cost = $\$75F + \$24(P_1 + P_2 + P_3 + P_4 + P_5)$

Constraints: For each hour, the available labor-hours must be at least equal to the required labor-hours:

$$F + P_1 \geq 10 \text{ (9 A.M. to 10 A.M. needs)}$$

$$
\begin{aligned}
F + P_1 + P_2 &\geq 12 (10\ A.M. \text{ to } 11\ A.M. \text{ needs}) \\
\frac{1}{2}F + P_1 + P_2 + P_3 &\geq 14 (11\ A.M. \text{ to noon needs}) \\
\frac{1}{2}F + P_1 + P_2 + P_3 + P_4 &\geq 16 (\text{noon to } 1\ P.M. \text{ needs}) \\
F + P_2 + P_3 + P_4 + P_5 &\geq 18 (1\ P.M. \text{ to } 2\ P.M. \text{ needs}) \\
F + P_3 + P_4 + P_5 &\geq 17 (2\ P.M. \text{ to } 3\ P.M. \text{ needs}) \\
F + P_4 + P_5 &\geq 15 (3\ P.M. \text{ to } 4\ P.M. \text{ needs}) \\
F + P_5 &\geq 10 (4\ P.M. \text{ to } 5\ P.M. \text{ needs})
\end{aligned}
$$

Only 12 full-time tellers are available, so:

$$F \leq 12$$

Part-time worker-hours cannot exceed 50% of total hours required each day, which is the sum of the tellers needed each hour:

$$4(P_1 + P_2 + P_3 + P_4 + P_5) \leq .50(10 + 12 + 14 + 16 + 18 + 17 + 15 + 10)$$

or:

$$
\begin{aligned}
4P_1 + 4P_2 + 4P_3 + 4P_4 + 4P_5 &\leq 0.50(112) \\
F, P_1, P_2, P_3, P_4, P_5 &\geq 0
\end{aligned}
$$

There are two alternative optimal schedules that Mexico City Bank can follow. The first is to employ only 10 full-time tellers ($F = 10$) and to start 7 part-timers at 10 a.m. ($P_2 = 7$), 2 part-timers at 11 a.m. and noon ($P_3 = 2$ and $P_4 = 2$), and 3 part-timers at 1 p.m. ($P_5 = 3$). No part-timers would begin at 9 a.m.

The second solution also employs 10 full-time tellers, but starts 6 part-timers at 9 a.m. ($P_1 = 6$), 1 part-timer at 10 a.m. ($P_2 = 1$), 2 part-timers at 11 a.m. and noon ($P_3 = 2$ and $P_4 = 2$), and 3 part-timers at 1 p.m. ($P_5 = 3$). The cost of either of these two policies is $1,086 per day.

INSIGHT It is not unusual for multiple optimal solutions to exist in large LP problems. In this case, it gives management the option of selecting, at the same cost, between schedules. To find an alternate optimal solution, you may have to enter the constraints in a different sequence.

LEARNING EXERCISE The bank decides to give part-time employees a raise to $7 per hour. Does the solution change? [Answer: Yes, cost = $1,142, $F = 10$, $P_1 = 6$, $P_2 = 1$, $P_3 = 2$, $P_4 = 5$, $P_5 = 0$.]

RELATED PROBLEM B.18

The Simplex Method of LP

Most real-world linear programming problems have more than two variables and thus are too complex for graphical solution. A procedure called the **simplex method** may be used to find the optimal solution to such problems. The simplex method is actually an algorithm (or a set of instructions) with which we examine corner points in a methodical fashion until we arrive at the best solution—highest profit or lowest cost. Computer programs (such as Excel OM and POM for Windows) and Excel spreadsheets are available to solve linear programming problems via the simplex method.

Simplex method

An algorithm for solving linear programming problems of all sizes.

For details regarding the algebraic steps of the simplex algorithm, see Tutorial 3 at our text Web site or refer to a management science textbook.[2]

[2]See, for example, Barry Render, Ralph M. Stair, and Michael Hanna, *Quantitative Analysis for Management*, 11th ed. (Pearson Education, Inc., Upper Saddle River, NJ, 2013): Chapters 7–9; or Raju Balakrishnan, Barry Render, and Ralph M. Stair, *Managerial Decision Modeling with Spreadsheets*, 3rd ed. (Pearson Education, Inc., Upper Saddle River, NJ, 2012): Chapters 2–4.