Definitions -

O.D.E. - ordinary differential equation

- has one independent variable, e.g. $t, x, \ldots$
- all derivitives are wrt the independent variable

P.D.E. - partial differential equation

- has multiple independent variable, e.g. $x, y, \ldots$
- derivitives are wrt multiple variables

- Order of the O.D.E. is the order of the highest order derivitive in the O.D.E.

- Solution of the O.D.E.
  - equates to the dependent variable and has no derivitives in it.

- Rate Problems -
  - involve a rate of change in some dependent variable wrt an independent variable, e.g. $\frac{dv}{dt}$, $\frac{di}{dt}$, $\frac{dv}{dx}$, ...
  - Types of rate problems: <u>population growth</u> or decay - could be <u>material</u>, or living organisms; physics types - changes in acceleration, speed, position, force, power, energy, ... * including electron flow (current), charge transfer, fluid transfer; Structural variations -

– slopes – contour – slope fields – graphical displays – direction fields –

– Basic types of O.D.E. solutions:

- Separation of Variables – manipulate and integrate
- Linear O.D.E. exact

– Linear O.D.E. : $y' + P(x)y = Q(x)$

  – no higher order powers of y or y' or functions of y or y'

  – Linear with Integrating Factor

  – Homogeneous equation → S. of V. solution –

Exact : $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ in $P(x,y) + Q(x,y)\dfrac{dy}{dx} = 0$

S. of. V. : $\dfrac{dy}{dx} = g(x) \cdot f(y)$

Linear : $\dfrac{dy}{dx} + P(x)y = Q(x)$

Homogeneous: $\dfrac{dy}{dx} = g\left(\dfrac{y}{x}\right)$

– O.D.E. may be more than one type.

i.e. $xy' = y \rightarrow \dfrac{dy}{dx} = \left(\dfrac{1}{x}\right)y$ (S. of. V)

$\rightarrow \dfrac{dy}{dx} + \left(-\dfrac{1}{x}\right)y = 0$ (linear ~~homogeneous~~)

$\rightarrow \dfrac{dy}{dx} = \dfrac{y}{x}$ (homogeneous)

~~Later we will examine other solution methods~~

- Numerical solutions : include Euler, Runge-Kutta, Series methods
- LaPlace
- LaPlace & Matrix Methods to solve systems of differential equations

– Initial Value Problems – The solution of the O.D.E. also satisfies ~~an~~ initial conditions

– The order of the differential equation determines the number of initial conditions in a particular solution.

– General solution – without initial conditions – multiple solutions possible

– Particular solution – with initial conditions – a single solution results from specific initial conditions.

$\frac{dx}{dt} = \dot{x}(t) = \dot{x} = Dx = x'$ = first order dirivitive

$x \frac{dy}{dx} = e^{xy}$ → 1st order O.D.E.

$y''' - 2xy' + y = \sin x$ → 3rd order O.D.E.

$\frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = \phi$ → 2nd order P.D.E.

$\frac{dy}{dx} = F(x,y)$ → general 1st order O.D.E. form

if $F(x,y) = g(x) f(y)$ then S. of V. –

manipulate and integrate →

– RATE IN – RATE OUT – STORAGE PROBLEM { ultracaps – batteries, thermal barrier, ...

EX. Salt tank –

Problem Statement :
- 20 kg salt dissolved in 5 kL of H_2O in tank
- brine of 0.03 kg salt per L enters tank at 25 L/min.
- brine drains from tank at the same rate.

- How much salt remains in the tank after ½ hour?

Solution : $y(t)$ is amount of salt after t minutes,
$y(0) = 20$ kg
We want to find $y(30)$.

$\frac{dy}{dt}$ is the rate of change of the amount of salt,

$\frac{dy}{dt}$ = (rate in) – (rate out)

$$\text{rate in} = \left(0.03 \frac{kg}{L}\right)\left(25 \frac{L}{min}\right) = 0.75 \frac{kg}{min}$$

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– The tank always contains 5000 L, so the concentration at time t is $\frac{y(t)}{5000}\left[\frac{kg}{L}\right]$.

– Brine flows out at $25\left[\frac{L}{min}\right] \rightarrow$

$$\text{rate out} = \left(\frac{y(t)}{5k}\left[\frac{kg}{L}\right]\right)\left(25\left[\frac{L}{min}\right]\right) = \frac{y(t)}{200}\left[\frac{kg}{min}\right]$$

$$\rightarrow \quad \frac{dy}{dt} = 0.75\ \frac{kg}{min} - \frac{y(t)}{200}\ \frac{kg}{min} = \frac{150 - y(t)}{200}\ \frac{kg}{min}$$

$$\rightarrow \quad S.\ of\ V. \rightarrow \quad \frac{dy}{dt} = \frac{150 - y(t)}{200} \quad \left[\frac{kg}{min}\right]$$

$$\rightarrow \quad \int \frac{dy}{150 - y(t)} = \int \frac{dt}{200}$$

$$\rightarrow \quad -\ln|150 - y(t)| = \frac{t}{200} + C$$

(multiply by −1)
(and apply i.c.)

$$y(0) = 20 \rightarrow \quad \ln|130| = C \rightarrow$$

$$\rightarrow \quad |150 - y(t)| = 130\, e^{-t/200}$$

{ $y(0)=20$ and right side is always positive, thus $|150 - y(t)| = 150 - y(t)$

(multiply by −1) → { put in standard form } →

$$\rightarrow \quad y(t) = 150 - 130\, e^{-t/200}$$

$e^{a+b} = e^a e^b$

→ after 30 minutes →

$$y(30) = 150 - 130\, e^{-30/200} \simeq 38.1\ kg$$

— RATE PROBLEM — Decay or Growth —
— bacteria (living organisms), materials (radioactive waste), system cooling or heating (unidirectional), interest compounding :

Common interest aaruing formula : $A(t) = A_0\left(1+\frac{i}{n}\right)^{nt}$

, t in years , A_0 is initial investment,
i is the interest rate , n is the number of times the interest is compounded annually,

If we let $n \to \infty$, then we compound interest continually, and

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{i}{n}\right)^{nt} \qquad \Rightarrow \left(1+\frac{i}{n}\right)^{nt} = \left[\left(1+\frac{i}{n}\right)^{\frac{n}{i}}\right]^{it}, \quad m = \frac{n}{i}$$

$$= A_0 e^{it} \qquad \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m} = e$$

→ If we differentiate this, we have the rate at which our investment increases,

$$\frac{dA(t)}{dt} = i A_0 e^{it} = i A(t)$$

which says the rate of increase is proportional to the interest rate and the size of the investment.

This O.D.E. can be solved with S.of.V. to determine the initial investment required at a given i, for a particular term of savings, required to reach a particular final amount of savings.

Problem 1.
H.W.#1: Solve this – show all work – With continuous

compounding, how many years do you need to save with \$1000 initial deposit, to reach a target \$2000 savings with 1.5% interest?

Problem. 2. EE's only:

Assume a series RL circuit with a DC battery source:

$$L\frac{di}{dt} + Ri = V \quad , \; i(0) = \phi$$

Find $i(t)$ using S. of. V. — Show all work.

Problem. 2. ME's only: Same problem with force, mass, ~~and~~ position, and velocity:

$$m\frac{dv}{dt} + bv = F \quad , \; v(\phi) = \phi$$

, m = mass, b = viscous friction coefficient, v = velocity, and F is force.

Let $L = m = \phi.1$, $R = b = \phi.2$, $V = F = 1\phi$

Exact O.D.E.:

$$P(x,y) + Q(x,y)\frac{dy}{dx} = \phi$$

is exact if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $P(x,y) = f_x(x,y)$ and $Q(x,y) = f_y(x,y)$

The solution is in the form $f(x,y) = C$

See sheets "Section 15.1" pgs 1093-1095

& "Integrating Factors" same Section pgs 1096-1097.

& B44 3.4 Predator-Prey Model; pg 25 - examples