$$\Gamma(u,v) = [\underbrace{\sin v \cos u}_{x}, \underbrace{\sin v \sin u}_{y}, \underbrace{\cos v}_{z}]^T$$

the parameterization $\Gamma(u,v) = [\sin v \cos u, \sin v \sin u, \cos v]^T$ on $A = \{(u,v) : 0 < u < 2\pi,\ 0 < v < \pi\}$ and

$$D\left(\frac{\partial\Gamma}{\partial u}(p_0), \frac{\partial\Gamma}{\partial v}(p_0)\right) = \left\| \frac{\partial\Gamma}{\partial u}(p_0) \times \frac{\partial\Gamma}{\partial v}(p_0) \right\|$$

$$= \left[\frac{\partial(y,z)}{\partial(u,v)}, -\frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right]^T_{p_0} = \sin v .$$

Exercise 78 If α and β are one-forms in variables r and p, then $\Gamma^*(\alpha \wedge \beta) = (\Gamma^*\alpha) \wedge (\Gamma^*\beta)$. Show that $\Gamma^*(dr \wedge dp) = \rho\, d\rho \wedge d\theta$ in polar coordinates.

18. Total Energy Function are Hamiltonian Functions

Energy is more than just a conserved quantity : it determines the equations of motion. Here we'll show how to derive these equations from the energy function and the kinetics of the phase space.

Consider a particle moving on a line, with position r and momentum p. If the particle has mass m and moves under the influence of a conservative force then the Total energy function, also known as the Hamiltonian function, for this particle is

$$H := \frac{1}{2m} p^2 + \mathcal{U}(r)$$

where $\mathcal{U} : \mathbb{R} \to \mathbb{R}$ is a potential energy function and the equation of motion is $\frac{d^2}{dt^2} r = \frac{d}{dr} \mathcal{U}(r)$.

Exercise 79 The Hamiltonian directly implies that

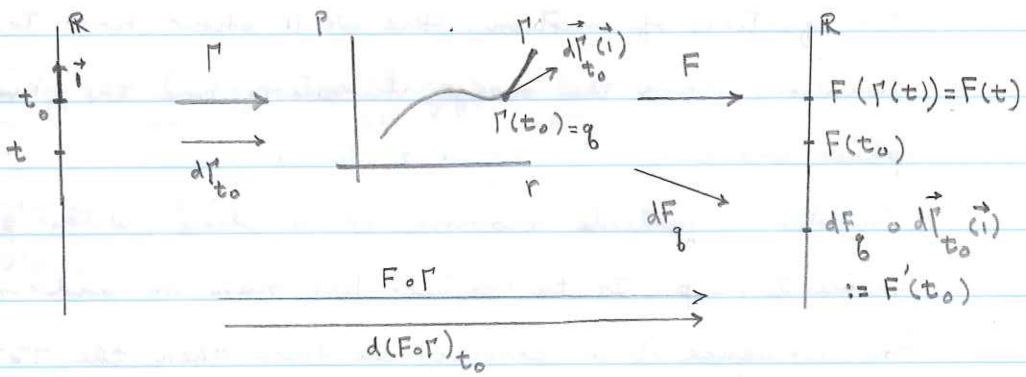
H does not depend on t. explicitly

a) Total Energy is conserved : $\frac{d}{dt} H = 0$

b) $\frac{dr}{dt} = \frac{\partial H}{\partial p}$ and $\frac{dp}{dt} = -\frac{\partial H}{\partial r}$. These equations are known as Hamilton's equations.

c) show that Hamilton's equations imply that H is conserved, and also imply that $m \frac{d^2}{dt^2} r = \mathcal{U}'(r)$.

Consider any function F defined on this phase space (such as temperature, kinetic energy, or total energy).



we compute :

$$[d(F\circ\Gamma)_{t_0}] = [dF_q][d\vec{\Gamma}_{t_0}] = \begin{bmatrix}\frac{\partial F}{\partial r} & \frac{\partial F}{\partial p}\end{bmatrix}_q \begin{bmatrix}\frac{dr}{dt}\\ \frac{dp}{dt}\end{bmatrix}_{t_0}$$

$$\Rightarrow \quad \frac{d}{dt}F(r,p) = \frac{dr}{dt}\frac{\partial F}{\partial r} + \frac{dp}{dt}\frac{\partial F}{\partial p}$$

Definition $\frac{d}{dt} = \frac{\partial H}{\partial p}\frac{\partial}{\partial r} - \frac{\partial H}{\partial r}\frac{\partial}{\partial p}$ is the Hamiltonian vector field and is denoted by X_H.

Let's write Hamilton's equations in the more formal language of symplectic geometry. The phase space is $\mathbb{R}^2$ with symplectic area form $\omega = dp\wedge dr$. The "gradient" 1-form of H is (see page 43 at the bottom!)

$dH = \frac{\partial H}{\partial r}dr + \frac{\partial H}{\partial p}dp$.

Exercise 80 show : $\omega(X_H, \overset{\text{anything}}{\cdot}) = -dH$

Hint : $dp\wedge dr(\vec{v},\vec{w}) = D([dp(\vec{v}), dp(\vec{w})]^T, [dr(\vec{v}), dr(\vec{w})]^T)$.

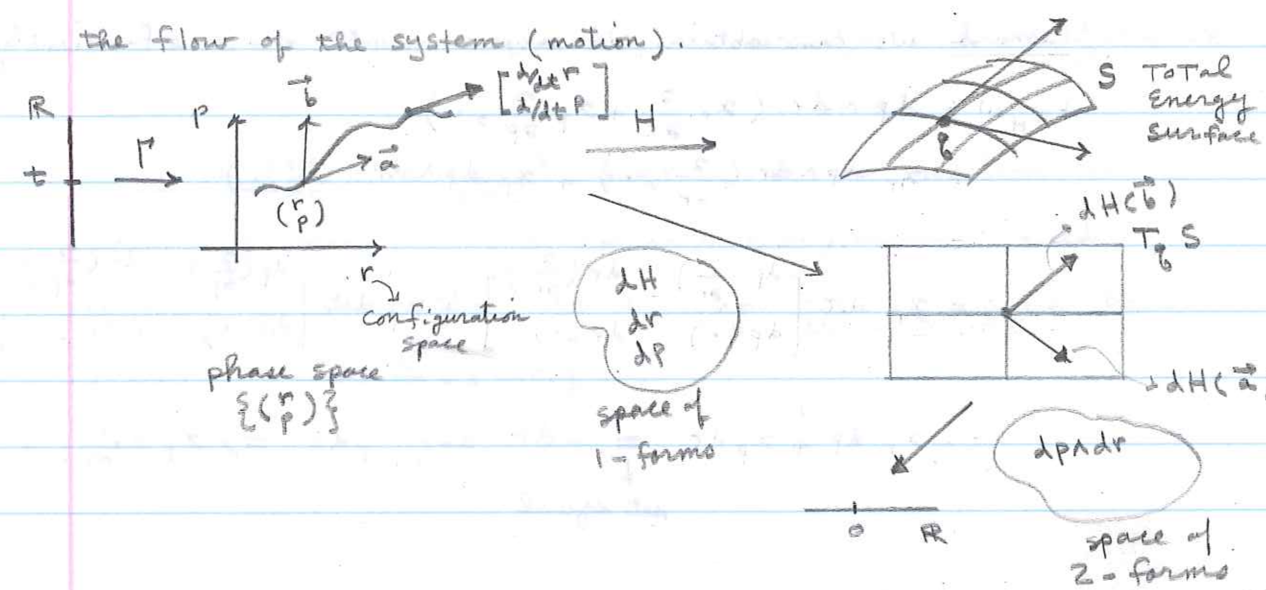
Exercise 81 show that $\omega(X_H, \cdot) = -dH \Rightarrow$ Hamilton's equations. Hint let $\cdot = \frac{\partial}{\partial r}$ or $\frac{\partial}{\partial p}$

Remark we have just showed that Hamilton's equations are equivalent to the equation $\omega(X_H,\cdot) = -dH$.

notation : $\iota_{X_H}\omega := \omega(X_H,\cdot)$; so that equation of ex. 80 and Hamilton's equations can be written as

$$\iota_{X_H}\omega = -dH .$$

Comment this equation says that it is the symplectic structure of phase space ω (kinematics), in conjunction with the H function (energy function), that determines the flow of the system (motion).



<u>Example</u> a free particle of mass m moving on the line is subject to no forces. Our physical intuition (due to Newton) says that any free particle will move with constant speed. The Hamiltonian function is $H = \frac{1}{2m}p^2$.

Recalling that $X_H := \frac{d}{dt}$ is the Hamiltonian vector field, say, $X_H = X_r(r,p)\frac{\partial}{\partial r} + X_p(r,p)\frac{\partial}{\partial p}$, then $\iota_{X_H}\omega$ applied to an arbitrary vector field $\vec{v} = v_r(r,p)\frac{\partial}{\partial r} + v_p(r,p)\frac{\partial}{\partial p}$, gives

$$\begin{aligned} \iota_{X_H}\omega(\vec{v}) &= dp\wedge dr\,(X_H, \vec{v}) \\ &= dp\wedge dr\left(X_r\frac{\partial}{\partial r} + X_p\frac{\partial}{\partial p},\ v_r\frac{\partial}{\partial r} + v_p\frac{\partial}{\partial p}\right) \\ &= X_p v_r - X_r v_p . \end{aligned}$$

On the other hand, $-dH(\vec{v}) = -\frac{1}{m}p\,dp(\vec{v}) = -\frac{1}{m}p\,v_p$. Then $\iota_{X_H}\omega(\vec{v}) = -dH(\vec{v}) \Rightarrow X_p(r,p) = 0$ and $X_r(r,p) = \frac{1}{m}p$

$\Rightarrow X_H := \frac{d}{dt} = \frac{1}{m}p\frac{\partial}{\partial r}$.

<u>Remark</u> we can obtain the above result more efficiently:

$$\begin{aligned} \iota_{X_H}\omega &= dp\wedge dr\left(X_r\frac{\partial}{\partial r} + X_p\frac{\partial}{\partial p}, \cdot\right) \\ &= X_r\, dp\wedge dr\left(\frac{\partial}{\partial r}, \cdot\right) + X_p\, dp\wedge dr\left(\frac{\partial}{\partial p}, \cdot\right) \\ &= X_r \det\begin{bmatrix} dp(\frac{\partial}{\partial r}) & dr(\frac{\partial}{\partial r}) \\ dp(\cdot) & dr(\cdot)\end{bmatrix} + X_p \det\begin{bmatrix} dp(\frac{\partial}{\partial p}) & dr(\frac{\partial}{\partial p}) \\ dp(\cdot) & dr(\cdot)\end{bmatrix}. \\ &= -X_r\, dp + X_p\, dr \underset{\text{set equal}}{=} -dH = -\frac{1}{m}p\,dp \Rightarrow X_r = \frac{1}{m}p . \end{aligned}$$

We now derive the equations of motion from the expression for X_H: on one hand, $\frac{d}{dt} = \frac{dr}{dt}\frac{\partial}{\partial r} + \frac{dp}{dt}\frac{\partial}{\partial p}$; on the other hand, $\frac{d}{dt} = X_H = \frac{1}{m}p\frac{\partial}{\partial r}$. Since $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial p}$ are linearly independent it follows that

$$\frac{d}{dt}r = \frac{1}{m}p \quad\text{and}\quad \frac{d}{dt}p = 0 \iff \frac{d^2}{dt^2}r = 0,$$

which says that a free particle moves with constant momentum.

<u>Example</u> a particle of mass m moves on a line and is subject to the force of a spring. The Newtonian equation of motion is $\frac{d^2}{dt^2}r = -kr$ where $k>0$ is the spring constant. The H. function is $H = \frac{1}{2m}p^2 + \frac{k}{2}r^2$ where $U(r) = \frac{1}{2}kr^2$ is the potential function for the spring. Notice that U increases with r.

<u>Exercise 82</u> Show: $\iota_{X_H}\omega = -dH \Rightarrow \frac{d}{dt}r = \frac{1}{m}p$ & $\frac{d}{dt}p = -kr$.

<u>Exercise 83</u> $H := \frac{1}{2m}p^2 + mgr$ is the H function for a particle of mass m in a constant gravitational field of strength g. Show that the equation $\frac{d}{dt} = X_H$ implies that $m\frac{d}{dt}r = p$ and $\frac{d}{dt}p = -mg$.

Example Magnetism in three-space

Because the magnetic field is not a conservative field it can not be encoded in the Hamiltonian. But it can be encoded in the 2-form

$$\omega := \sum_{i=1}^{3} dp_i \wedge dr_i + q\,[B_1\, dr_2 \wedge dr_3 + B_2\, dr_3 \wedge dr_1 + B_3\, dr_1 \wedge dr_2],$$

where q is the charge and $\vec{B} = [B_1, B_2, B_3]^T$ is the magnetic field vector.

If all other forces acting on the particle are conservative, they can be combined into one potential function $U : \mathbb{R}^3 \to \mathbb{R}$ so that the equation of motion for the particle of mass m is

$$m \frac{d^2}{dt^2} \vec{r} = \frac{q}{m} \vec{p}^T \times \vec{B} - \vec{\nabla} U$$

where $\vec{r} = [r_1, r_2, r_3]^T$ is the configuration space and $\{(\vec{r}, \vec{p} = [p_1, p_2, p_3]^T)\}$ the phase space for the particles motion. We choose the Hamiltonian

$$H(\vec{r}, \vec{p}) := \frac{1}{2m} \vec{p}^{\,2} + U(\vec{r}),$$

where $\vec{p}^{\,2} = \vec{p}^T \vec{p} = p_1^2 + p_2^2 + p_3^2$.

Exercise 84 Show that $\iota_{X_H} \omega = -dH \implies$ the Hamiltonian vector field is given by

$$X_H = \frac{1}{m} \sum_{i=1}^{3} p_i \frac{\partial}{\partial r_i}$$

$$+ \left[\frac{q}{m} (p_2 B_3 - p_3 B_2) - \frac{\partial U}{\partial r_1} \right] \frac{\partial}{\partial p_1}$$

$$+ \left[\frac{q}{m} (p_3 B_1 - p_1 B_3) - \frac{\partial U}{\partial r_2} \right] \frac{\partial}{\partial p_2}$$

$$+ \left[\frac{q}{m} (p_1 B_2 - p_2 B_1) - \frac{\partial U}{\partial r_3} \right] \frac{\partial}{\partial p_3};$$

finally write down the equations of motion for the particle.

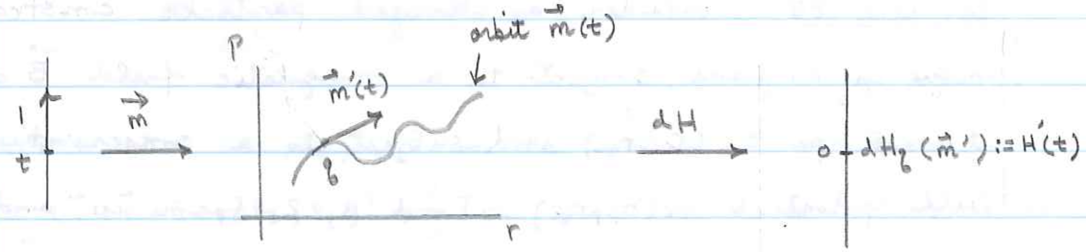
Exercise 85 Consider a charged particle constrained to move in a plane, subject to a magnetic field $\vec{B} = [B_1, B_2]^T$ at each point (r_1, r_2) and subject to a conservative force with potential $U(r_1, r_2)$. Find a 2-form ω and a function H which show that this motion is a Hamiltonian system; finally write down the equations of motion for the particle.

Application to Energy conservation

An orbit of the system is a set of all points in phase space that the system passes through during one particular motion. It is a basic feature of a Hamiltonian motion (flow) that its orbits are subsets of level sets (contours) of H. To show this recall that the Hamiltonian vector field "X_H" represents the operation $\frac{d}{dt}$. Then given a solution $\vec{m}(t) = [r(t), p(t)]^T$ to the equations of motion,

$$X_H H = \frac{d}{dt} H(t) := d(H \circ \vec{m})_t (1)$$
$$= dH_q \circ d\vec{m}_t (1)$$
$$= dH(\vec{m}'(t) \cdot 1)$$
$$= dH(\tfrac{d}{dt}\vec{m}(t) \cdot 1)$$

See Figure on p.70

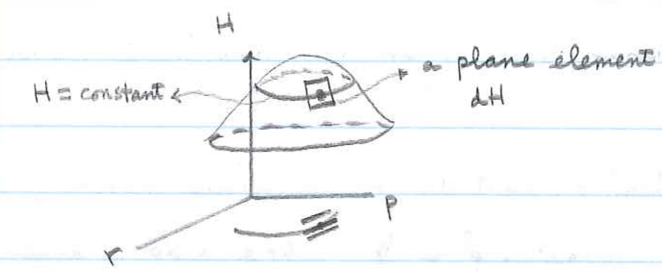


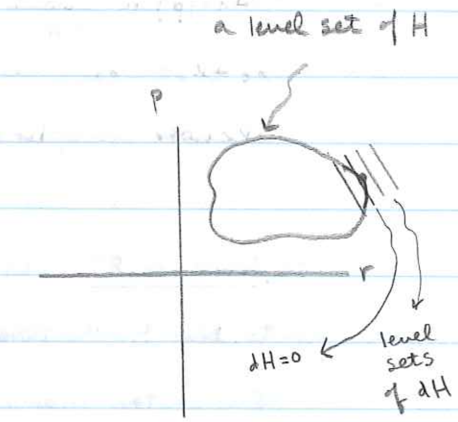
In "disembodied" version

$$X_H H = dH(X_H) = -\omega(X_H, X_H) = 0$$

$\Rightarrow \quad 0 = X_H H = \frac{d}{dt} H$ i.e. the Hamiltonian flow X_H preserves the Hamiltonian function.

This says that the orbits of the Hamiltonian flow are contained to "curves" of constant energy:



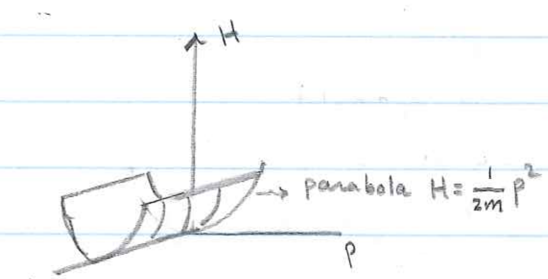


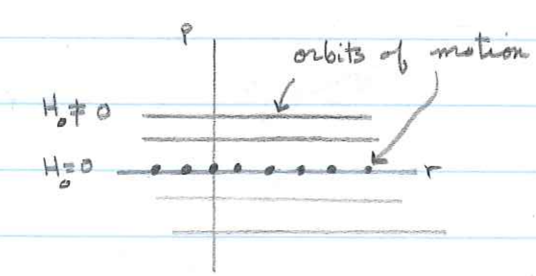
Remark we can get information about orbits (solutions) of the differential equations of motions (usually hard to solve) by solving the algebraic equation $H =$ constant, which is easier to solve!

Example a free particle on a line has its orbits inside sets of the form $H(r,p) = \frac{1}{2m} p^2 =$ constant H_0. By Hamilton's equations

$$\frac{dr}{dt} = \frac{\partial H}{\partial p} = \frac{1}{m} p = \pm\sqrt{\frac{2H_0}{m}} \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial r} = 0 .$$

$\Rightarrow \quad r(t) = \pm\sqrt{\frac{2H_0}{m}}\, t + r(0) = \frac{1}{m} t\, p(0) + r(0)$, and $p(t) = p(0)$.





Exercise 86 for the spring mass system $H(r,p) = \frac{1}{2m} p^2 + \frac{1}{2} k r^2$. Check explicitly that $X_H H = 0$, so that each solution (orbit) must lie entirely inside one ellipse in phase space.

Exercise 87 Consider a particle on a line moving according to the Hamiltonian $H = \frac{1}{2m} p^2 - \frac{k}{r}$, where $k > 0$ is a constant. This is the Hamiltonian for a two-body problem where the system has no angular momentum (the relative displacement of the bodies is parallel to their relative momentum). Use the equations of motion and the conservation of the Hamiltonian to show that on an orbit where $H = H_0$ (a constant) we have

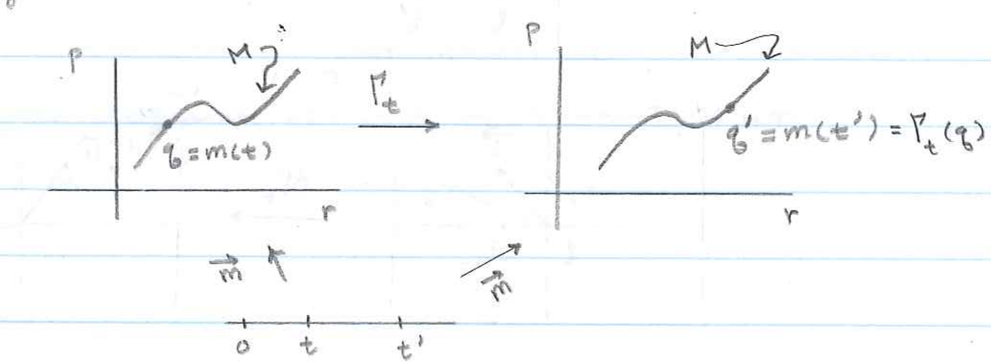
$$\frac{d}{dt} r = \pm \sqrt{\frac{2}{m}\left(H_0 + \frac{k}{r}\right)} .$$

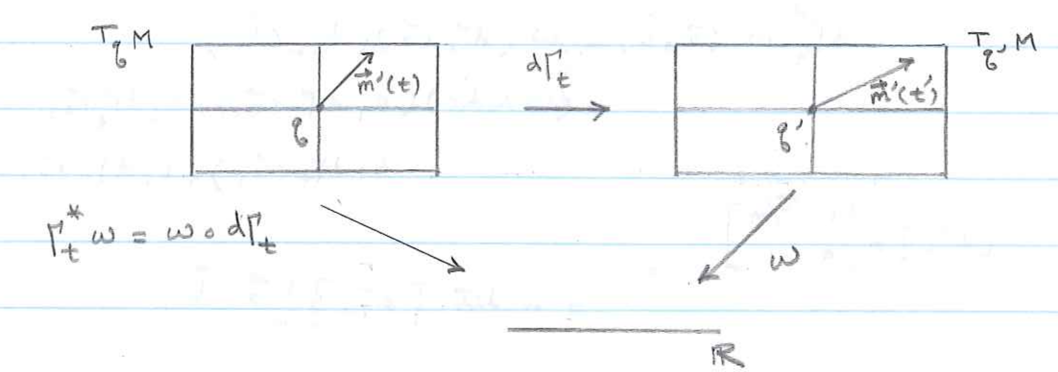
Solve this equation (by separation of variables) and interpret the solutions physically.

Another fundamental theorem of Hamiltonian mechanics says that the H. flow preserves the (symplectic) form ω. In symbols, $X_H \omega = 0$. Proving this result is beyond the scope here, but we may see this in another way.

Define the Hamiltonian flow $\Gamma_t : M \to M$ on the phase space M, where $t \in \mathbb{R}$; and require that the "time"-dependent function $\Gamma_t(m)$ solve the equations of a Hamiltonian motion: i) $\frac{d}{dt} \Gamma_t(m) = X_H \Gamma_t(m)$ and $\Gamma_0(m) = m$.

Remark Here we require that the operator for motion "$\frac{d}{dt}$" be the Hamitonian vector field $X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial r} - \frac{\partial H}{\partial r}\frac{\partial}{\partial p}$ corresponding to the H. function H.



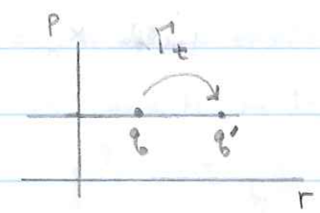


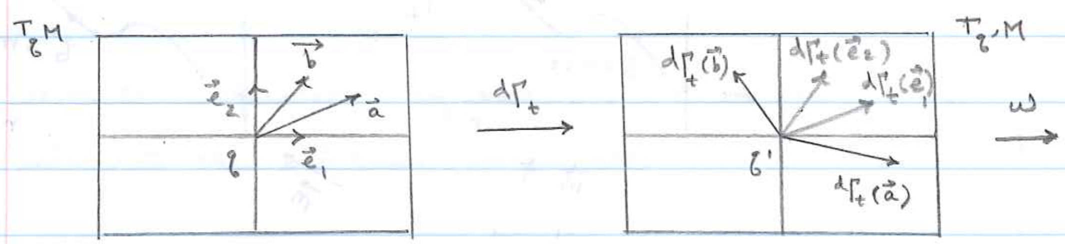
<u>theorem</u> Let (M, ω, H) be a Hamiltonian system. Then for each $t \in \mathbb{R}$, $\Gamma_t^* \omega = \omega$.

<u>Remark</u> this states that locally the form ω is preserved under the flow or the form is not affected by the flow.

<u>Example</u> Reconsider the free particle moving on a line (example page 77) with $\omega = dp \wedge dr$ and

$$q' = \Gamma_t \begin{pmatrix} r \\ p \end{pmatrix} = \begin{bmatrix} 1 & \frac{1}{m} t \\ 0 & 1 \end{bmatrix} \begin{pmatrix} r \\ p \end{pmatrix} :$$





$$d\Gamma_t^* \omega (\vec{a}, \vec{b}) = \omega (d\Gamma_t(\vec{a}), d\Gamma_t(\vec{b}))$$
$$= (dp \wedge dr)(a_1 d\Gamma_t(\vec{e}_1) + a_2 d\Gamma_t(\vec{e}_2),$$
$$b_1 d\Gamma_t(\vec{e}_1) + b_2 d\Gamma_t(\vec{e}_2))$$

$$[d\Gamma_t] = \begin{bmatrix} 1 & t/m \\ 0 & 1 \end{bmatrix}$$

$$= - \det [d\Gamma_t] [\vec{a} \,\vdots\, \vec{b}]$$

$$= -(1) \det [\vec{a} \,\vdots\, \vec{b}] = \omega(\vec{a}, \vec{b}).$$

<u>Example</u> the flow $\Gamma_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\Gamma_t \begin{pmatrix} r \\ p \end{pmatrix} = \begin{pmatrix} re^t \\ pe^t \end{pmatrix}$ does <u>not</u> preserve the symplectic form $\omega = dp \wedge dr$.

For any $t \in \mathbb{R}$,

$$\Gamma_t^* \omega = \Gamma_t^* (dp \wedge dr) = \Gamma_t^* dp \wedge \Gamma_t^* dr$$
$$= d\left(p \circ \Gamma_t \begin{pmatrix} r \\ p \end{pmatrix}\right) \wedge d\left(r \circ \Gamma_t \begin{pmatrix} r \\ p \end{pmatrix}\right)$$
$$= d(pe^t) \wedge d(re^t)$$
$$= e^{2t} dp \wedge dr$$
$$= e^{2t} \omega .$$

<u>Remark</u> a non-Hamiltonian system does not preserve areas of parallelograms spanned by tangent vectors under the flow Γ_t.