Math 121G Exploration #1 Writing mathematical arguments -- demonstrating equivalence.

Warm-Up

Graph the following piece-wise function. Include two tables of values. Use an appropriate domain and range.

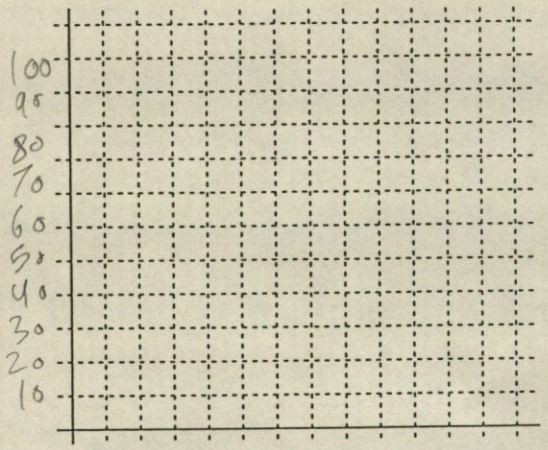
$$f(x)=\begin{cases}50+3x & \text{if } x\le 10\\ 40-2x & \text{if } x>10\end{cases}$$

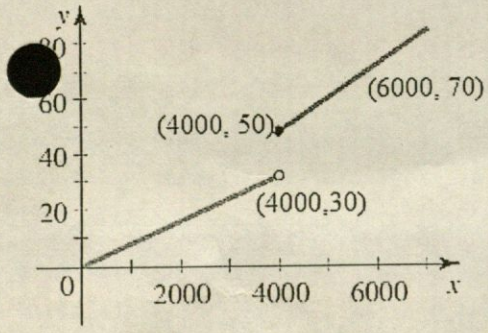
$x \le 10$

x	y
10	80
7	71
5	65
0	50

$x > 10$

x	y
11	18
10	16
13	14
14	10





Two students modeled the graph above differently. Demonstrate that the two models are equivalent.

Student #1

$$f(x)=\begin{cases}\dfrac{30}{4000}x & \text{if } x<4000\\ \dfrac{20}{2000}x+10 & \text{if } x\ge 4000\end{cases}$$

Student #2

$$f(x)=\begin{cases}.0075x & \text{if } x<4000\\ .01(x-4000)+50 & \text{if } x\ge 4000\end{cases}$$

$\frac{30}{4000} = .0075$, So first rule is equivalent

1. Demonstrate that a linear equation given in the form, $Ax+By+C=0$ will always have a slope of $\frac{-A}{B}$ (unless B is zero and the slope is undefined) and a y-intercept of $\frac{-C}{B}$. Include an example.

Rules for mathematical Arguments:
(1) each line is equalent
(2) one mathmatical step or process per line.
(3) verbal explanation of what is happening

$Ax+By+C=0$

$Ax+By=-C$

$\frac{By}{B}=\frac{-C-Ax}{B}$

$y=\frac{-C}{B}-\frac{A}{B}x$

* In this equation $-\frac{A}{B}$ is the slope and $\frac{-C}{B}$ is the y-intercept

2. Demonstrate that a non-vertical line with a slope of m and that passes through the point (x_o, y_o) will always have a vertical intercept of $y_o - mx_o$. Include an example.

$y-y_o=m(x-x_o)$

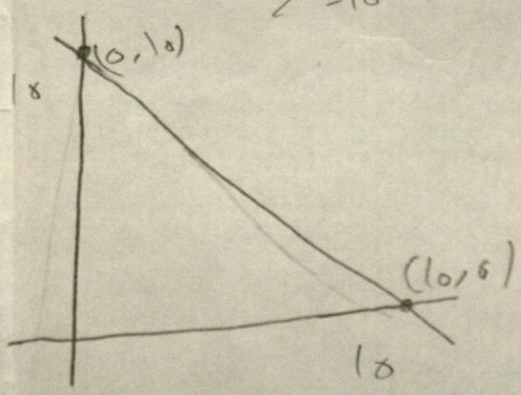
$y=y_o=mx-my_o$

$y=mx-mx_o+y_o$

b

3. The same line can have many different equations in point-slope or general form, but only one equation in slope-intercept form. Explain why this is true, including examples.

$x=10-x$



$y=mx+b$ ← slope-intercept

$y-y_o=m(x-x_o)$ ← point slope

$ax+by=c$ ← general form

$(y-10)=0-1(x-10)$

$(y-10)=-1(x-0)$

$x+y=10$

$2x+2y=20$

$3x+3y=30$

Math 121G Exploration #3 Working with Functions – Composed and Inverse, Shifted and Stretched

1. Consider the two functions, $f(x)=100-30x$ and $g(x)=15x+25$.

   a. Find $H(x)=f(g(x))$.

   $H(x) = 100 - 30(15x+25)$

   $H(x) \quad 100 - 450x - 750$

   $H(x) = -450x - 650$

   b. Find $f^{-1}(x)$ and $g^{-1}(x)$.

   $f(x) = 100 - 30x$

   $y = 100 - 30x$

   $y = -100 - 30x$

   $\frac{-100+y}{-30} = x \Rightarrow y = \frac{-100+x}{-30}$ simplify

   c. Find $H^{-1}(x)$.

   $H^{-1}(x) = \left[\frac{x-100}{-30} - 25\right] \div 15$ $\quad H^{-1}(x) = \frac{x+650}{-450}$

   d. Draw two flow diagrams that shows that $H(x)$ and $H^{-1}(x)$ are inverse functions.

   $x \rightarrow \times 15 \rightarrow +25 \rightarrow \times -30 \rightarrow +100 \in H(x)$

   $H^{-1}(x) \Rightarrow \div 15 \leftarrow -25 \leftarrow \div -30 \leftarrow -100 \leftarrow x$

   e. Verify that $H(x)$ and $H^{-1}(x)$ are really inverse functions by composing them.

   $H(x) = -650 + 450$

   $H^{-1}(x) = \frac{x+650}{-450} = \frac{-x-650}{450}$

   $H(H^{-1}(x)) = -650 - 450\left[\frac{x+650}{-450}\right]$

   $= -650 + x + 650$

   $= x$

2. Frank claims that to find the inverse of a composed function you simply compose the two inverse functions. Is he correct? Explain. You may refer to the example you just worked, but your argument should be general and hold for any two pairs of functions and their composition.

$x \to f(x) \longrightarrow g(x) \Rightarrow g(f(x)) = h(x)$

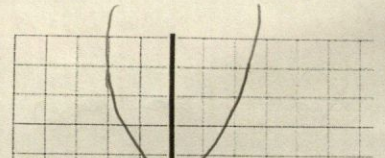
inverse

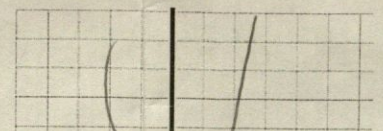
$f^{-1}(g^{-1}(x)) = f^{-1}(x) \Leftarrow g^{-1}(x) \Leftarrow x = h^{-1}(x)$

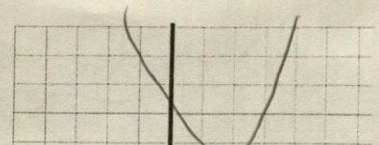
3. Sketch the graphs of these functions. Explain how your graphs demonstrate the concepts of stretching and shifting.

a. $f(x) = x^2;\ g(x) = x^2 - 3;\ h(x) = (x-2)^2$

x	y
0	0
1	1
2	4
-1	1
-2	4







b. Now graph the three inverse relations – These aren't one-to-one functions, so their inverses will not be functions, they will violate the vertical line test. Write each rule above each graph. The first one is done for you.

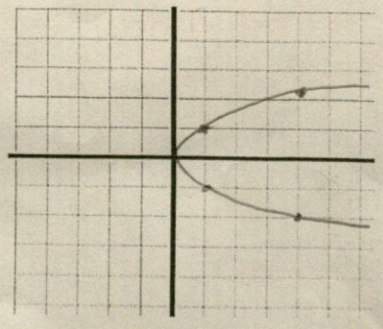
$y = \pm\sqrt{x}$

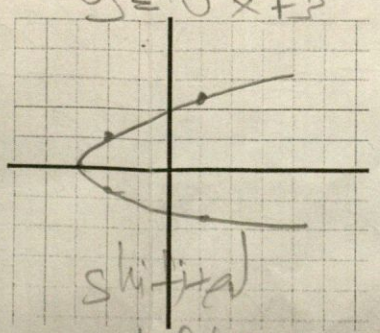
$y = x^2 - 3$

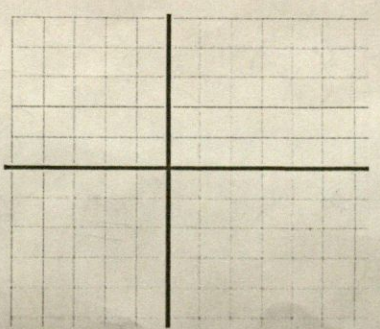
$\sqrt{y+3} = \sqrt{x^2} \Leftarrow x$

$y = \sqrt{x+3}$

x	y
0	0
1	1
4	2







4. Sketch the graphs of these functions. Explain how your graphs demonstrate the concepts of stretching and shifting. Write the domain and range for each graph.

a. $f(x)=\sqrt{x};\ g(x)=\sqrt{x-2};\ h(x)=\sqrt{x+4}-3$

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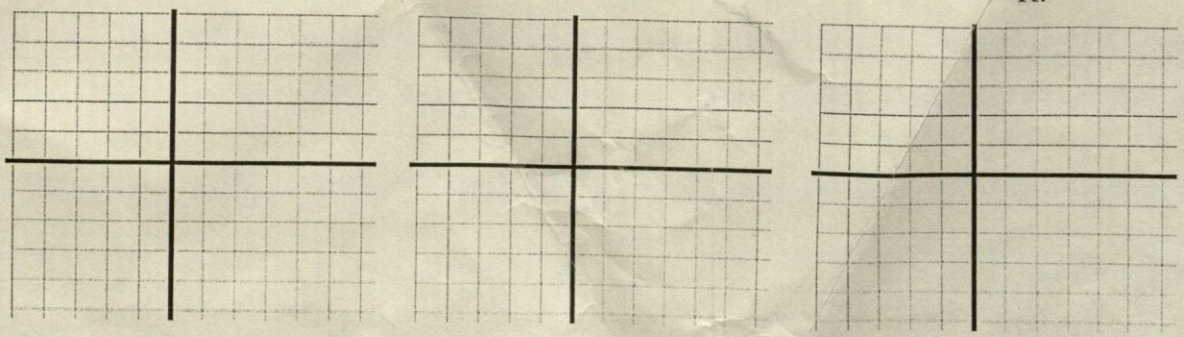
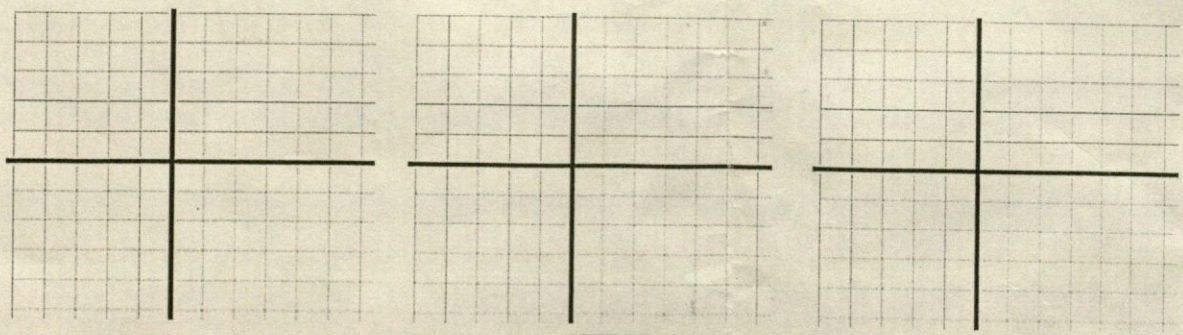


b. Now graph the three inverse functions – The original functions have domain and range restrictions. Write each inverse function above each graph, include the appropriate domain and range.

$y=x^2$

D: $[0,\infty)$ R: $[0,\infty)$



5. The functions, $f(x)=(x-2)^3+4$ and $g(x)=\sqrt[3]{(x-4)}+2$ are inverse functions. Demonstrate that this is true by composing them.

Exploration 4 – Equivalent Exponential Models

1. Oscar invests \$5000 in a Certificate of Deposit (CD) at his bank. The CD pays 3.2% compounded monthly.

   a. Write the equation for the function, $A(t)$, that gives the value of the CD after t years.

   $A(t) = 5000(1 + .032)^t$

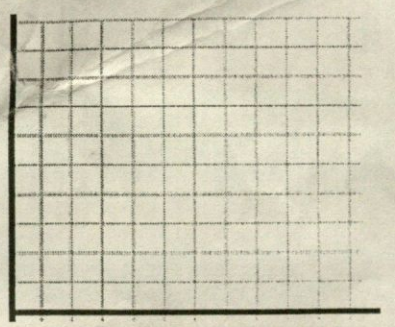
   b. Find the annual percent yield for the CD.

   3.2%

   c. Write an equivalent equation for the value of the CD after t years that is in the form, $A(t) = Ca^t$

   d. How long does it take for the CD to double in value? Solve graphically.

   e. Sketch the graph of $A(t)$. Justify your choice of domain and range.



2. The number of bacteria in a sample doubles every 7 hours. Explain why the two models below are equivalent.

   $P(t) = P_o(\sqrt[7]{2})^t$ and $P(t) = P_o(2)^{\frac{t}{7}}$ where t is the number of hours since the population was first measured. $= 2^{\frac{t}{7}}$

3. Explain why the two models, $f(x) = 10\left(\frac{1}{3}\right)^x$ and $g(x) = 10(3)^{-x}$ are equivalent.

Using scientific notation, we express the world population as 6.5×10^9. The positive exponent 9 indicates that the decimal point should be moved nine places to the *right*:

$$6.5 \times 10^9 = 6{,}500{,}000{,}000$$

Move decimal point 9 places to the right

Similarly, we can express the mass of a hydrogen atom as 1.66×10^{-24} g. The negative exponent -24 indicates that the decimal point should be moved 24 places to the left:

$$1.66 \times 10^{-24} = 0.\ 000000000000000000000001\ 66$$

Move decimal point 24 places to the left

4. Explain why you move the decimal place to the left and to the right when converting from scientific notation to standard notation.

Exploration 5 – Log and Exponential Formulas

The Richter Scale

The earth moves all the time. Small earthquakes happen daily, most are indiscernible. The Richter Scale is a common log scale, with the magnitude of an earthquake given by the formula $M = \log\frac{I}{S}$ where I is the intensity of the earthquake and S is the intensity of a 'standard' earthquake, so small that it would count as zero on the Richter scale.

1. Use the formula to show that an earthquake with an intensity of S has a magnitude of zero.

2. Use the magnitude formula to show that an earthquake with an intensity that is 100 times larger than S has a magnitude of 2.

3. Use the magnitude formula to show that an earthquake with a magnitude of 5 has an intensity that is 100,000 times larger than S.

4. Why is the Richter Scale a 'power of ten' scale?

5. If one earthquake has a magnitude of 7.2 and a second earthquake has an intensity that is 18 times larger than the first, what is the magnitude of the second earthquake?

6. Solve the magnitude formula for I and write the inverse formula.

7. Compose the original and the inverse formula. Compose both ways and simplify each result. Explain your result.

8. Use your formula for intensity to show that an earthquake with a magnitude of 6.5 is 100 times more intense than an earthquake with a magnitude of 4.5.

Scientists also calculate the seismic energy of earthquakes. Two formulas are given below. M is the magnitude and E is the seismic energy in ergs. Use these formulas to answer the following questions.

$$E = 10^{11.8+1.5M} \quad M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$

9. How much seismic energy is released by a magnitude 5.2 earthquake?

10. What is the magnitude of an earthquake that releases 10^{23} ergs of seismic energy?

11. Compose the two formulas and simplify the result. Are they inverse functions? How do you know?

Exploration 6 – Newton's Law of Cooling

Newton's Law of Cooling states that the temperature T of a cooling object is modeled by $T = A + (I - A)e^{-kt}$ where t is the time since the object began cooling, I is the initial temperature of the object, A is the ambient temperature (the temperature of the surroundings), and k is a constant that depends on the type of object, movement of air or surrounding fluids, and the unit of time being used.

1. A pot of chili is 212°F and must cool to 70 °F before being placed in the refrigerator. Assume that the ambient temperature is 65°F and the heat transfer coefficient is $k = 2.895$ for hours.

a) Find a model for the temperature of the pot of chili after t hours.

b) What is the temperature of the chili after 30 minutes?

c) How long will it take the pot of chili to reach a temperature of 70 °F? Show work.

c) Find a model for the temperature of the pot of chili t hours after it is placed in a 35 °F refrigerator. Assume it was 70 °F when it went in.

d) How long will it take the pot of chili to reach a temperature of 40 °F?

2. A roasted turkey is taken from the oven with its temperature has reached 185 °F and placed on a table in a room where the temperature is 70 °F. After one hour, the temperature of the turkey has fallen to 125 °F.

a) What is the heat transfer coefficient?

b) Find a model for the temperature of the turkey after t minutes.

c) Use your model to predict the temperature of the turkey after 2 hours.

3. A 25 pound turkey comes out of the freezer at a temperature of 22 °F and is placed on the counter in a 70 °F room.

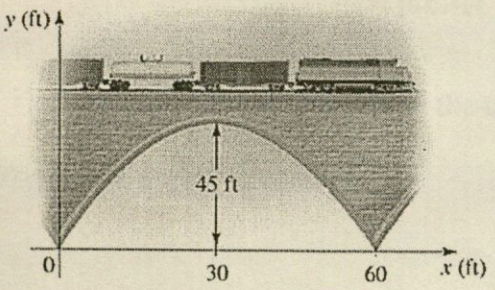
a) If it takes 27 hours to reach a temperature of 60 °F when sitting in a 70 °F room, what is the value of the heat transfer coefficient, k?

b) Find a model for the turkey's temperature t hours after being taken out of the freezer.

c) What was the temperature of the turkey 10 hours after being taken out of the freezer?

Exploration 7 – Applications of Quadratic Functions

48. Parabolic Truss Bridge The shape of a suspension bridge can be turned upside down to create a bridge where each supporting truss is in the shape of a parabola. The truss of the bridge shown in the figure is 45 feet high at its highest point, and the two points where it touches the ground are 60 feet apart, as shown in the figure. Find a quadratic function that models the parabolic truss.

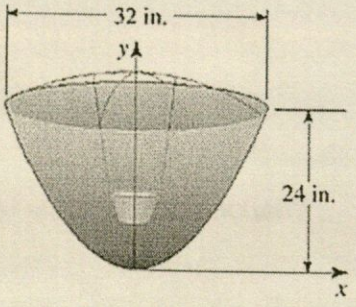


Where is the origin on the graph?

Do you know vertex or x-intercepts?

What form of equation do you want to start with?

45. Parabolic Cooker *Solar cookers* are parabolic reflectors that concentrate the light and heat of sunlight at one spot, where a pot of food can be placed to cook. They are sometimes sold in camping stores and are also increasingly used in impoverished areas of the world where cooking fuel is scarce. The figure shows a parabolic bowl-shaped cooker in cross-section; the width is 32 inches, and the depth is 24 inches. Find a squaring function that models the shape of the cooker.



Where do you want to place the origin on the graph?

What form of equation do you want to start with?

23. Height of a Ball If a ball is thrown directly upward with a velocity of 12 m/s, its height (in meters) after t seconds is given by $y = 12t - 4.9t^2$. What is the maximum height attained by the ball, and after how many seconds is that height attained?

this a vertex or x-intercept problem?

28. Dividing a Pen A rancher with 750 feet of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle (see the figure).

(a) Find a function A that models the total area of the four pens.

(b) Find the largest possible total area of the four pens and the dimensions of that area.

Draw the picture and define your variables.

Write the perimeter constraint.

Solve the perimeter constraint for y.

What is the formula for area?

Substitute your expression for y for the

letter y in the area formula.

Now find the vertex of A(x).

24. Path of a Ball A ball is thrown across a playing field from a height of 5 ft above the ground at an angle of 45° to the horizontal and at a speed of 20 ft/s. It can be deduced from physical principles that the path of the ball is modeled by the function

$$y = -\frac{32}{(20)^2}x^2 + x + 5$$

where x is the distance in feet that the ball has traveled horizontally and y is the height in feet. What is the maximum height attained by the ball, and at what horizontal distance does this occur?

Is this a vertex or x-intercept problem?

34. Maximizing Profit A community bird-watching society makes and sells simple bird feeders to raise money for its conservation activities. The materials for each feeder cost $6, and the society sells an average of 20 feeders per week at a price of $10 each. The society has been considering raising the price, so it conducts a survey and finds that for every dollar increase, it loses 2 sales per week.

(a) Find a function P that models weekly profit in terms of price per feeder.

(b) What price should the society charge for each feeder to maximize profits? What is the maximum weekly profit?

Profits equal revenues (money in to the cash register) minus costs (money out of the cash register).

What is the cost function, C(x)?

To find Revenues, you multiply the price, p(x), times the number sold, x. First you must find the linear price function. What information do you have about the relationship between price and the number sold?

In p(x), what is the input and what is the output?

What is the Revenue function?

What is the profit function?

Find the vertex of the profit function and interpret your result.

For the Group assignment!

48. Maximizing Revenue Miriam sells souvenir caps to tourists in a beach town. She observes that if she charges x dollars per cap, then she sells about caps per day. (Notice that the higher her price, the fewer caps she sells.)

(a) Explain why her revenue per day is given by the function $R(x) = x(48 - 1.6x)$.

(b) What price should she charge so that her revenue is as large as possible?

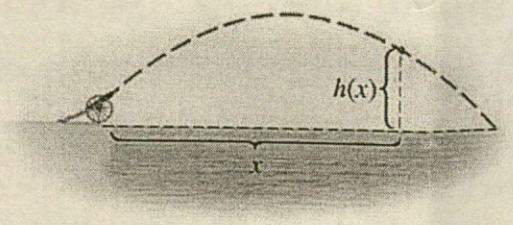
6. A cannonball that is fired out to sea from a shore battery follows a parabolic trajectory given by the graph of the equation

$$h(x) = 10x - 0.01x^2$$

where $h(x)$ is the height of the cannonball above the water when it has traveled a horizontal distance of x feet.

(a) What is the maximum height that the cannonball reaches?

(b) How far does the cannonball travel horizontally before splashing into the water?



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Exploration 8 –Interpreting Rational Functions

. WebAssign Problem:

43. Flight of a Rocket Suppose a rocket is fired upward from the surface of the earth with an initial velocity v (measured in meters per second). Then the maximum height h (in meters) reached by the rocket is given by the function

$$h(v) = \frac{Rv^2}{2gR - v^2}$$

where the radius of the earth R is 6.4×10^6 m and the acceleration due to gravity g is 9.8 m/s^2.

(a) If the initial velocity of the rocket is 10,400 m/s, what is the maximum height reached by the rocket?

(b) Use a graphing calculator to draw a graph of the function $y = h(v)$. (Note that y and v must both be positive, so the viewing rectangle need not contain negative values.) What does the vertical asymptote represent physically?

2. For a person with sensitive skin, the amount of time T (in hours) the person can be exposed to the sun with minimal burning can be modeled by $T = \frac{0.37s + 23.8}{s}$ where s is the Sunsor Scale reading and the domain is $0 \le s \le 120$. The Sunsor Scale is based on the level of intensity of UVB rays.

a. Find the amounts of time a person with sensitive skin can be exposed to the sun with minimal burning when s = 10, s = 25, and s = 100.

b. What would the horizontal asymptote of the graph be? _____ What would the horizontal asymptote represent?

3. Psychologists have developed a mathematical model to predict memory performance as a function of the number of trials, n, of a certain task. Consider the learning curve $P = \frac{0.5+0.9(n-1)}{1+0.9(n-1)}$ where P is the fraction of correct responses after n trials.

a. Complete the table below.

n	1	2	3	4	7	10	15
P	.5	.74	.82	.86	.98	.95	.96

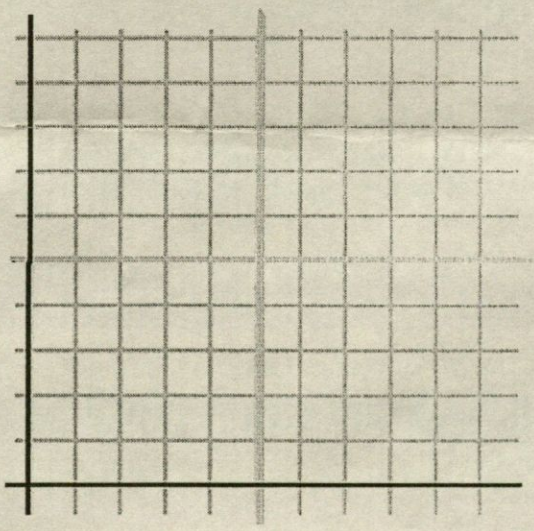
b. What is the horizontal asymptote for this equation? _____ What does it tell you about the performance of the task?

4. The cost of supplying recycling bins to p% of the population of a small city is given by $C = \frac{25000p}{100-p}$.

a. Graph the function.

b. Find the cost of supplying bins to 10%, 25% and 90% of the population.

c. Can you supply bins to 100% of the population? _____ Explain.



For the Group Assignment:

The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is modeled by $N = \frac{20(5+3t)}{1+0.04t}$ where t is the number of years since the deer were first introduced.

a. How large will the population be after 10 years?

b. What is the horizontal asymptote? _____ What does the horizontal asymptote tell you about the deer?

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