DEFINITION: A ***double tracing*** is a closed walk that traverses every edge exactly twice. A double tracing is ***bidirectional*** if every edge is used once in each of its two directions.

Proposition 6.1.4. *Every connected graph has a bidirectional double tracing. In a tree, every double tracing is bidirectional.* ♢ (Exercises)

EXERCISES for Section 6.1

In Exercises 6.1.1 through 6.1.4, determine which graphs in the given graph family are eulerian.

6.1.**1**[S] The complete graph K_n.

6.1.**2** The complete bipartite graph $K_{m,n}$.

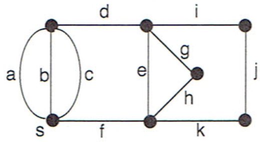
6.1.**3** The n-vertex wheel W_n.

6.1.**4** The hypercube graph Q_n.

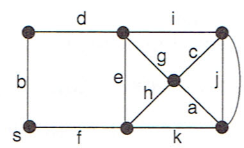
6.1.**5**[S] Which platonic graphs (§1.2) are eulerian?

In Exercises 6.1.6 through 6.1.9, apply Algorithm 6.1.1 to construct an eulerian tour of the given graph. Begin the construction at vertex s.

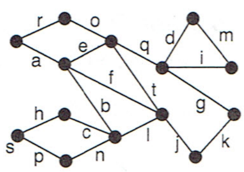
6.1.**6**



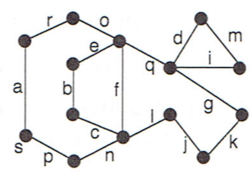
6.1.**7**[S]



6.1.**8**



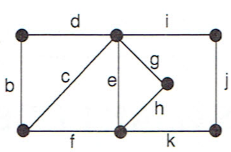
6.1.**9**



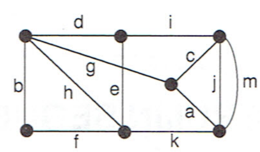
6.1.**10**[S] Use the strategy of the second half of the proof of Theorem 6.1.1 to design a variation of Algorithm 6.1.1 that constructs an open eulerian trail in any graph that has exactly two vertices of odd degree.

In Exercises 6.1.11 through 6.1.14, use the modified version of Algorithm 6.1.1 from Exercise 6.1.10 to construct an open eulerian trail in the given graph.

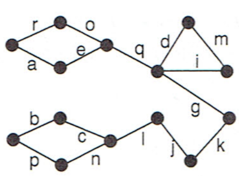
6.1.**11**



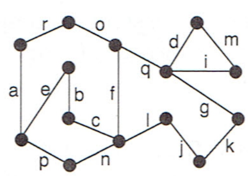
6.1.**12**



6.1.**13**[S]



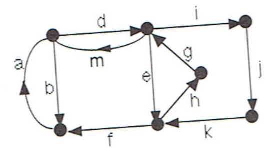
6.1.**14**



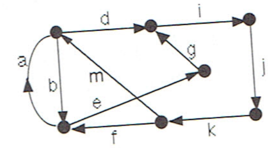
6.1.**15**[S] Design a variation of Algorithm 6.1.1 that constructs an eulerian tour of any eulerian digraph.

In Exercises 6.1.16 through 6.1.19, use an appropriate modification of Algorithm 6.1.1 to construct an eulerian tour or open eulerian trail in the given digraph.

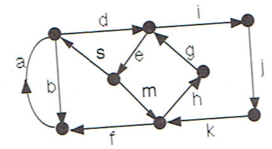
6.1.16



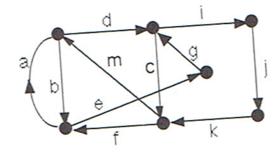
6.1.17



6.1.18[S]



6.1.19



6.1.**20** Referring to the initial step of Algorithm 6.1.1, why is it always possible to construct a closed trail, regardless of the choice of starting vertex?

6.1.**21**[S] Referring to the while-loop in Algorithm 6.1.1, why is it always possible to find a vertex on trail T that is incident on an unused edge?

6.1.**22** Prove Theorem 6.1.2. (Hint: See proof of eulerian-graph characterization in §4.5.)

6.1.**23** Prove Theorem 6.1.3. (Hint: Use Theorem 6.1.2.)

6.1.**24** Prove Proposition 6.1.0.

6.1.**25**[S] Prove that if a simple graph G is eulerian, then its line graph $L(G)$ (§1.2) is eulerian.

6.1.**26** Prove or disprove: The line graph of *any* eulerian graph is eulerian.

6.1.**27**[S] Prove or disprove: If the line graph of a graph G is eulerian, then G is eulerian.

6.1.**28** Prove that if a connected graph G has $2k$ vertices of odd degree, then there is a set of k edge-disjoint trails that use all the edges of G.

6.1.**29** [*Computer Project*] Implement Algorithm 6.1.1 and run the program on each of the graphs in Exercises 6.1.6 through 6.1.9.

6.1.**30** [*Computer Project*] Implement a modified version of Algorithm 6.1.1 so that it finds an open eulerian trail in a graph that has exactly two vertices of odd degree. Run the program on the graphs in Exercises 6.1.11 through 6.1.14.

6.2 DEBRUIJN SEQUENCES AND POSTMAN PROBLEMS

DEFINITION: A bitstring of length 2^n is called a $(2, n)$-***deBruijn sequence*** if each of the 2^n possible bitstrings of length n occurs *exactly once* as a substring, where wraparound is allowed.

Example 6.2.1: The following four bitstrings are $(2, n)$-deBruijn sequences for the cases $n = 1, 2, 3, 4$, respectively.

01 0110 01110100 0000100110101111

6.2.8[S] Construct the deBruijn digraph $D_{3,3}$ and use it to construct a $(3,3)$-deBruijn sequence.

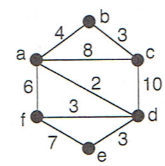
6.2.9 Show that the line graph $L(D_{3,2})$ of the deBruijn digraph $D_{3,2}$ is the deBruijn digraph $D_{3,3}$.

6.2.10 Prove that $L(D_{p,n}) = D_{p,n+1}$.

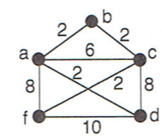
6.2.11 Which vertices of the deBruijn digraph $D_{2,n}$ have self-loops? Justify your answer.

In Exercises 6.2.12 through 6.2.15, apply Algorithm 6.2.2 to find a minimum-weight postman tour for the given weighted graph. Determine whether the solution is unique.

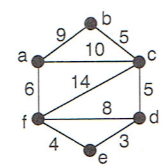
6.2.12



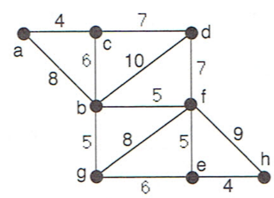
6.2.13



6.2.14[S]



6.2.15



Exercises 6.2.16 through 6.2.18 refer to Algorithm 6.2.2.

6.2.16[S] Prove that a perfect matching of K is guaranteed to exist.

6.2.17 Prove that G^* is eulerian.

6.2.18 Show how the optimality of the postman tour follows from having chosen a minimum-weight perfect matching of edges that correspond to shortest paths in G.

6.2.19 Use a modified version of Algorithm 6.2.2 to solve the example problem in Application 6.2.3.

6.2.20 Solve the problem posed in Application 6.2.4.

In Exercises 6.2.21 through 6.2.24, use the method of Application 6.2.5 to find an RNA chain whose G- and UC-fragments are as given.

6.2.21 G-fragments: CCG, G, UCCG, AAAG;
UC-fragments: GGAAAG, GU, C, C, C, C.

6.2.22[S] G-fragments: CUG, CAAG, G, UC;
UC-fragments: C, C, U, AAGC, GGU.

6.2.23 G-fragments: G, UCG, G, G, UU;
UC-fragments: GGGU, U, GU, C.

6.2.24 G-fragments: G, G, CC, CUG, G;
UC-fragments: GGGU, U, C, GC.

6.2.25 Referring to Application 6.2.5, construct a chain that contains no internal subfragments and is not irreducible.

EXERCISES for Section 6.3

In Exercises 6.3.1 through 6.3.4, determine which graphs in the given graph family are hamiltonian.

6.3.1 The complete graph K_n.

6.3.2[S] The complete bipartite graph $K_{m,n}$.

6.3.3 The n-vertex wheel W_n.

6.3.4 Trees on n vertices.

6.3.5 Which platonic graphs are hamiltonian?

In Exercises 6.3.6 through 6.3.10, draw the specified graph or prove that it does not exist.

6.3.6[S] An 8-vertex simple graph with more than 8 edges that is both eulerian and hamiltonian.

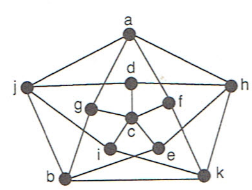
6.3.7 An 8-vertex simple graph with more than 8 edges that is eulerian but not hamiltonian.

6.3.8 An 8-vertex simple graph with more than 8 edges that is hamiltonian but not eulerian.

6.3.9 An 8-vertex simple hamiltonian graph that does not satisfy the conditions of Ore's theorem.

6.3.10 A 6-vertex simple graph with 10 edges that is not hamiltonian.

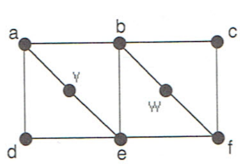
6.3.11[S] Prove that the **Grötzsch graph**, shown below, is hamiltonian.



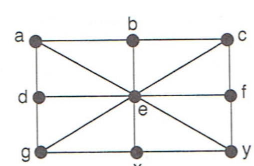
6.3.12 Prove that a bipartite graph that is hamiltonian must have an even number of vertices.

In Exercises 6.3.13 through 6.3.18, either construct a hamiltonian cycle in the given graph, or prove that the graph is not hamiltonian.

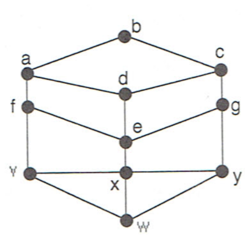
6.3.13[S]



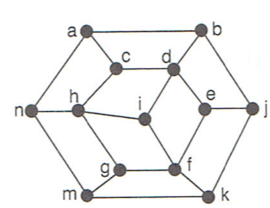
6.3.14



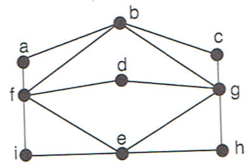
6.3.15



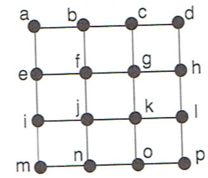
6.3.16



6.3.17



6.3.18[S]



6.3.19 Show that the Petersen graph is not hamiltonian.

6.3.20 One version of the Icosian Game was to find a hamiltonian cycle that started with a given five letters (see Figure 6.3.1). Find all hamiltonian cycles that begin with the letters BCPNM.

6.3.21[S] Characterize the graphs for which the following properties hold.

a. a depth-first search produces a hamiltonian path, regardless of the starting vertex.

b. a breadth-first search produces a hamiltonian path, regardless of the starting vertex.

DEFINITION: A ***knight's tour*** of a chessboard is a sequence of knight moves that visits each square exactly once and returns to its starting square with one more move.

6.3.22 The problem of determining whether a knight's tour exists actually predates the work of Hamilton. Pose the knight's tour problem as one of determining whether a certain graph is hamiltonian.

6.3.23 Show that if the requirement in Dirac's result (Corollary 6.3) is relaxed to "$deg(v) \geq \frac{n-1}{2}$" (from "$deg(v) \geq \frac{n}{2}$"), then the assertion of the theorem is false.

6.3.24[S] Show that the sufficient condition in Ore's Theorem 6.2 is not a necessary condition, by giving an example of a hamiltonian graph that does not satisfy the conditions of the theorem.

6.3.25 Consider the complete graph K_n with vertices labeled $1, 2, \ldots, n$.

a. Find the number of different hamiltonian cycles. Two cycles that differ only in where they start and end should be counted as the same cycle.

b. Find the number of hamiltonian paths from vertex 1 to vertex 2.

c. Find the number of open hamiltonian paths.

6.3.26 Prove or disprove each of the following statements.

a. There exists a 6-vertex eulerian graph that is not hamiltonian.

b. There exists a 6-vertex hamiltonian graph that is not eulerian.

6.3.27 Show that for any odd prime n, the edges of K_n can be partitioned into $\frac{n-1}{2}$ edge-disjoint hamiltonian cycles. (Hint: Arrange the vertices $1, 2, \ldots, n$ around a circle.)

6.3.28 Suppose that 19 world leaders are to dine together at a circular table during a conference. It is desired that each leader sit next to a pair of different leaders for each dinner. How many consecutive dinners can be scheduled? (Hint: See the preceding exercise.)

6.3.29 A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the 27 $1 \times 1 \times 1$ subcubes. If the mouse starts at one corner and always moves on to an uneaten subcube, can it finish at the center of the cube?

6.5.5 Two eulerian graphs A and B are amalgamated across a subgraph C. Prove that the resulting graph G is eulerian if and only if the subgraph C is eulerian.

6.5.6 Decide whether the join $2K_1 + K_{1,4}$ is hamiltonian.

6.5.7 Decide whether the join $3K_1 + K_{1,4}$ is hamiltonian.

6.5.8 Decide whether $K_{3,4}$ is hamiltonian.

6.5.9 Give an example of two connected graphs, each with at least two vertices, whose join is non-hamiltonian.

6.5.10 Draw a hamiltonian graph H such that two disjoint copies of H can be amalgamated across K_3, so that the result is non-hamiltonian. Explain why the result is non-hamiltonian.

6.5.11 Show that the graph of Figure 6.5.2(a) has a hamiltonian path but no hamiltonian cycle.

6.5.12 Show that the graph of Figure 6.5.2(b) has a hamiltonian path but no hamiltonian cycle.

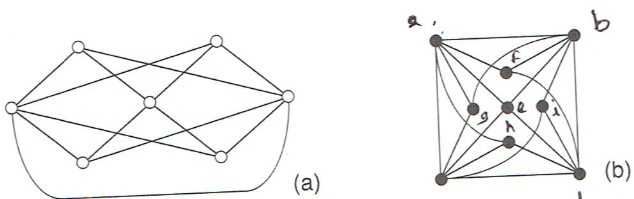


Figure 6.5.2

6.5.13 Decide whether the graph in Figure 6.5.3(a) is hamiltonian.

6.5.14 Decide whether the graph in Figure 6.5.3(b) is hamiltonian.

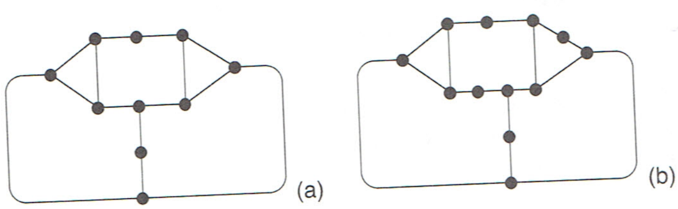


Figure 6.5.3

6.5.15 Suppose that two non-adjacent vertices are chosen in a hamiltonian graph H, and two disjoint copies of H are amalgamated across this vertex pair. Give examples to show that the resulting graph can be hamiltonian or non-hamiltonian.

6.5.16 Give a pair of graphs one of which is hamiltonian, the other non-hamiltonian, with the same number of vertices, such that all the vertex-deleted subgraphs of both graphs are non-hamiltonian. This would establish the non-reconstructibility of the hamiltonian property.

GLOSSARY

deadhead path in a postman tour: a path that is retraced.