CHAPTER SIX

Probability

PROBABILITY PLAYS A key role in statistics, serving as the building block for much of the underlying theory. Much statistical methodology involves the deductive statistics we first encountered in Chapter 1, which begins with assumed population characteristics and establishes the likelihood for various possible sample results. For example, probability will answer the question: "If a robot welder misses .1% of its targets, what is the probability it will still successfully complete a series of 20 spot welds?" Different approaches to deductive statistics are taken, depending on the type of population assumed.

Probability theory in turn forms the basis for inductive or inferential statistics, such as estimating strain, predicting curing time, determining whether to accept or reject a batch of components, or answering such design questions as "How much redundancy is needed in system communication?"

6–1 Fundamental Concepts of Probability

Probability theory has two basic building blocks. The more fundamental is the generating mechanism, called the **random experiment**, that gives rise to uncertain outcomes. Selecting one part for inspection from an incoming shipment is a random experiment having two elements of uncertainty: which particular item gets picked and the quality of that part. The second structural element of probability theory consists of the random experiment's outcomes, referred to as **events**. During a quality-control inspection, the usual events of interest for tested items are "good" and "defective."

Elementary Events and the Sample Space

A preliminary step in a probability evaluation is to catalog the events that might arise from the random experiment. Such a listing is made up of **elementary events**, which are the most detailed events of interest. A complete listing provides us with the random experiment's **sample space**.

For example, if we are interested only in the particular *upside showing face* from a coin toss, then we would have

$$\text{Coin toss sample space} = \{\text{head, tail}\} \qquad \text{(showing face)}$$

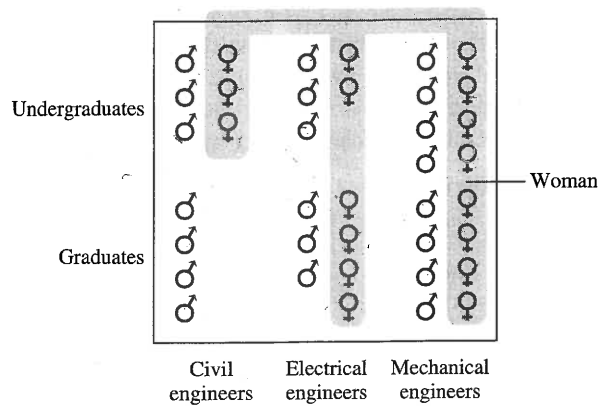


Figure 6–1
Sample space for random selection of a Tau Beta Pi member.

Alternatively, we might care only about the number of complete *revolutions*, and the same random experiment would yield an entirely different sample space:

$$\text{Coin toss sample space} = \{0, 1, 2, 3, \ldots\} \qquad \text{(revolutions)}$$

A sample space might be portrayed as a list, as above, or in some other convenient form. Sometimes it is conceptually helpful to use a picture, as in Figure 6–1, in which biological symbols are used to represent individual members of a Tau Beta Pi chapter. The random experiment is the selection of one member by lottery. Each person is an elementary event.

Event Sets

The immediate concern when finding a probability is properly identifying the event. This may be achieved by listing all the elementary events that give rise to the desired event. Such a collection is referred to as an **event set**.

Consider again the Tau Beta Pi lottery. A list of the names of female members represents the event set for the event "woman."

$$\text{Women} = \{\text{Ann, Barbara, Betty, Cheryl, Diane}, \ldots\}$$

This same event set might be pictured. Consider the tinted area in Figure 6–1.

Basic Definitions of Probability

The most common probability measure expresses the *long-run frequency* for an event occurring in many repeated random experiments. If a perfectly balanced coin is tossed *many times* without bias toward either side, we should obtain a head in about half the tosses. Thus, the long-run frequency of "head" is .50. We may then assume that .50 is the probability of "head," expressed symbolically as

$$\Pr[\text{head}] = .50$$

This value is an **objective probability**, and there should be no disagreement about how to find its value.

Objective probabilities can often be found, as above, through deductive reasoning alone. We don't actually have to toss the coin to reach the answer since we have no reason to believe that the head side will show any more or less than half the time.

Many commonly encountered random experiments involve elementary events that are *equally likely*. In such cases the probability for an event can be deduced directly.

PROBABILITY WHEN ELEMENTARY EVENTS ARE EQUALLY LIKELY

$$\text{Pr[event]} = \frac{\text{Size of event set}}{\text{Size of sample space}}$$

To apply the above "count-and-divide" procedure we need to know only (1) how many elementary events constitute the event set in question and (2) the total number of possibilities. For example, we know that "head" is the 1 and only elementary event in the event set for that face, that it is equally likely to occur versus "tail," and that these 2 elementary events constitute the sample space for the showing side from a coin toss. Thus,

$$\text{Pr[head]} = \frac{1}{2}$$

Consider again the Tau Beta Pi lottery summarized in Figure 6–1. We can apply this procedure to determine the probability that a woman will be selected. There are 17 women (that is, "woman" has an event set containing 17 elementary events, so that its size is 17). The sample space is made up of 38 equally likely elementary events, and

$$\text{Pr[woman]} = \frac{17}{38} = .447$$

Certain and Impossible Events

The basic definitions of probability express the value as a fraction because an objective probability is a long-run frequency. As such, it must be a value between 0 and 1, inclusive. Events having the extreme probabilities provide two interesting cases.

An outcome having zero probability is called an **impossible event**, since it cannot occur. Returning to the lottery, we find that the empty set

$$\text{Female graduate civil engineer} = \{\quad\} = \emptyset$$

applies because there was no student (elementary event) in the above category. The event "female graduate civil engineer" is an impossible lottery event. Applying the basic definition for equally likely elementary events, we have

$$\text{Pr [female graduate civil engineer]} = \frac{0}{38} = 0$$

All impossible events have zero probability.

As further illustration, we consider *mutually exclusive* events, only one of which may occur. That makes it impossible for them to occur jointly. For example, consider a part which may be classified as satisfactory or unsatisfactory, two mutually exclusive possibilities. Then for a single part, it follows

$$\text{Pr[satisfactory } \textit{and} \text{ unsatisfactory]} = 0$$

Likewise a new satellite will either work as intended or fail. These outcomes are mutually exclusive, and their simultaneous joint occurrence is an impossible event, so that

$$\Pr[\text{work } \textit{and} \text{ fail}] = 0$$

An outcome bound to occur is a **certain event** and has a probability of 1. Returning to the lottery, all possible names to be drawn are those of students, who constitute the entire sample space. Hence,

$$\Pr[\text{student is selected}] = \frac{38}{38} = 1$$

In tossing a coin, just two faces, "head" and "tail," are possible. (Should the coin land on its edge or be lost, the random experiment would be incomplete and another toss would have to be made.) Thus,

$$\Pr[\text{head } \textit{or} \text{ tail}] = \frac{2}{2} = 1$$

Logically Deduced Probabilities

Example: Quantum Mechanical Explanation of Photon Filtering

One of the findings of quantum mechanics is that a photon can exist in an ambiguous state until it is measured. Thus, the measured state might be an uncertain event for which a probability might be found. Consider an ideally polarized film and the way in which it filters light that is incident on it at a right angle. Light that is polarized in parallel to the transmission axis is certain to pass through the film, while light that is polarized perpendicular to that axis is certain to be blocked.

Suppose that the light is polarized at some angle θ to the film's transmission axis. The probability that any particular photon will be transmitted is computed from

$$\Pr[\text{transmission}] = \cos\theta$$

Thus, since $\cos(0°) = 1$, parallel light is *certain* to be transmitted, and since $\cos(90°) = 0$, it is *impossible* for perpendicular light to be transmitted. For light polarized at $\theta = 45°$,

$$\Pr[\text{transmission with } \theta = 45°] = \cos(45°) = .70711$$

Source: Abner Shimony, "The Reality of the Quantum World," *Scientific American*, January 1988, pp. 46–53.

Finding Probabilities from Experimentation

It may be difficult or impossible to logically deduce probabilities for some events. Returning once more to the coin toss, suppose that we take a fair coin and make it lopsided by dropping a few blobs of solder on one face. We would no longer be able to establish the probability value by reasoning alone, and Pr[head] might be a number such as .37 or .59. Only by actually tossing the coin many times, and keeping records of what sides show, could we finally arrive at an estimate of the proper probability value.

Problems

6–1 Consider the Tau Beta Pi lottery with the sample space shown in Figure 6–1. Find probabilities for the following events:

(a) man (c) civil engineer
(b) graduate (d) mechanical engineer

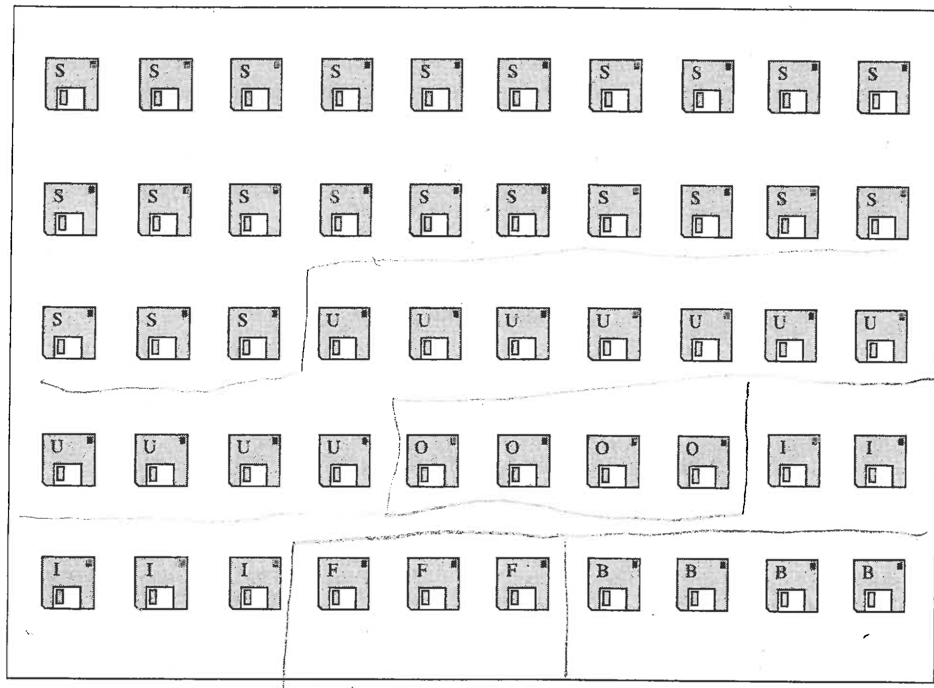
6–2 Explain how you would go about finding the probabilities for the following events:

(a) A vessel bursts before a pressure of 5,000 psi is applied. It is one of many production items.
(b) One of the defective items from a batch of known size having a known percentage of defectives is selected at random.
(c) A yet-to-be-fabricated prototype will fail.
(d) Some member of your graduating class receives a job offer paying in excess of $50,000 per year.

6–3 Consider the population of 50 floppy disks in Figure 6–2, with the quality state given for each disk. One disk is selected at random.

(a) Find the following quality state probabilities.
(1) Pr[S] (2) Pr[U] (3) Pr[O] (4) Pr[I] (5) Pr[F] (6) Pr[B]

Figure 6–2
Population of disks as defined by their quality states.



Quality codes: S–Satisfactory for receiving any data
U–Unformatted but good
I–Incomplete with some bad tracks
O–Occupied with some data
F–Damaged wafer
B–Broken housing

(b) One disk is selected at random. Find the following probabilities.
   (1) workable (S, U, O, or I)
   (2) unworkable (F or B)
   (3) ready to receive small data set (S, O, or I)
   (4) reformatting required to receive maximum capacity (U or O)

6–4 A pilot plan has yielded chemical batches categorized as follows:

	Unusable	Usable
Low in impurities	3	24
High in impurities	21	7

Assuming this experience is representative of full-scale operation, determine estimated probabilities that a production batch will be

(a) low in impurities
(b) high in impurities
(c) unusable
(d) usable
(e) both low in impurities and unusable
(f) both high in impurities and unusable
(g) both low in impurities and usable
(h) both high in impurities and usable

6–5 Four students toss a lopsided coin. The following results are obtained:

	Number of Occurrences				Total
	a	*b*	*c*	*d*	*e*
Heads	6	7	24	49	86
Tails	4	8	18	37	67
Total	10	15	42	86	153

Determine each student's estimated probability for "head." Then use their combined experience to estimate the probability of "head."

6–6 Consider a deck of 52 playing cards, from which a card is randomly selected. Sketch the sample space, letting each card be represented by a dot on a grid that has one row for each suit and one column for each denomination. Identify the event sets for the following as areas on your sketch, and calculate the respective probabilities.

(a) ten (b) heart (c) black (d) face card (e) two or three

6–7 Consider the random experiment of tossing a pair of six-sided dice (a red and a white one). The elementary events are the number of dots (1 through 6) appearing on the up sides of each die. Each outcome may be represented as an ordered pair. For example, a 3 for the red die and a 5 for the white die is (3, 5). Give a complete listing of the sample space. Then establish probabilities for each possible *sum* of the dots on the showing faces.

6–8 Consider a photon arriving at a right angle to a filter film. Letting θ represent the polarization angle relative to the transmission axis, find the probability ($\cos \theta$) that the photon will be transmitted when

(a) $\theta = 30°$ (b) $\theta = 60°$ (c) $\theta = 1°$ (d) $\theta = 89°$

6–2 Probabilities for Compound Events

Much like electric circuits formed by combining components, probabilities may be found for complex, *compound events* by combining the probabilities for other events, called *component events*. There are two basic kinds of compound events in which the components are viewed in terms of their union (*or*) or their intersection (*and*). Probability theory allows us to compute probabilities for compound events by arithmetically combining the probabilities of individual components. These procedures are based on either the addition law or the multiplication law.

Applying the Basic Definition

Before we describe these laws, we must emphasize that every compound event is itself an outcome for which the basic definitions of probability apply. When elementary events are equally likely, it is usually possible to compute Pr[*A or B*] or Pr[*C and D*] by simply counting possibilities and dividing by the total number of elementary events.

For example, consider the student lottery discussed earlier and shown again in Figure 6–3. We can use the count-and-divide procedure to find the probability that a randomly selected student is either an undergraduate or an electrical engineering major:

$$\Pr[\text{UG } or \text{ EE}] = \frac{26}{38}$$

(We make sure to count each undergraduate electrical engineer just once.) Likewise, we may apply the basic definitions to determine the probability for the joint occurrence of the same events:

$$\Pr[\text{UG } and \text{ EE}] = \frac{5}{38}$$

Probabilities for joint events are usually referred to as **joint probabilities**.

But in some applications we may not be able to perform these types of calculations—because the elementary events are not equally likely (as with a lopsided coin or

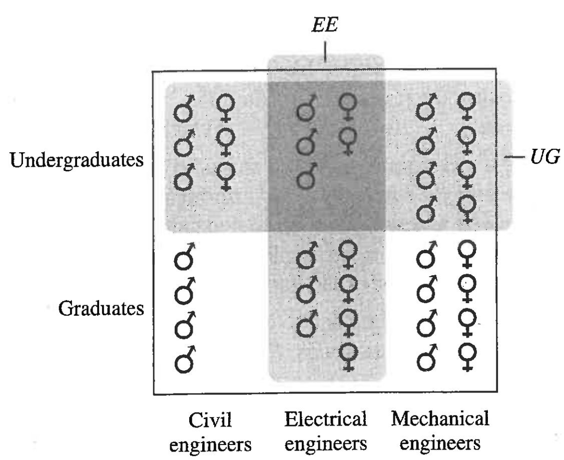


Figure 6–3
Sample space for random selection of a Tau Beta Pi member.

a crooked die cube), because we just don't have enough information, or because we cannot catalog the entire sample space. In such cases we might "synthesize" a compound event's probability by arithmetically manipulating the probabilities for its components (when these values are known).

The Addition Law

When a compound event is the union (*or*) of several components, the **addition law** allows us to find its probability by adding together the component event probabilities. There are two versions of this law. We begin with the simpler form.

ADDITION LAW FOR
MUTUALLY EXCLUSIVE EVENTS

$$\Pr[A \text{ or } B] = \Pr[A] + \Pr[B]$$

Illustration: *Power Cell Failure Probabilities*

As an illustration, suppose we know that the primary causes for failure of a space power cell are (1) solar radiation, (2) launch vibration, (3) material weakness, or (4) collision. For any particular failure, these causes may be considered as mutually exclusive events, since there can be only one primary reason for failing. Previous testing indicates that

$$\begin{aligned} \Pr[\text{solar radiation}] &= .10 \\ \Pr[\text{launch vibration}] &= .35 \\ \Pr[\text{material weakness}] &= .19 \\ \Pr[\text{collision}] &= .36 \end{aligned}$$

Causes (2) and (4) are mechanical accidents. To find the probability of this compound event, we apply the addition law:

$$\begin{aligned} \Pr[\text{mechanical accident}] &= \Pr[\text{launch vibration or collision}] \\ &= \Pr[\text{launch vibration}] + \Pr[\text{collision}] \\ &= .35 + .36 = .71 \end{aligned}$$

The addition law for mutually exclusive events extends to any number of components. For instance, consider the probabilities in Table 6–1 for the number of power cell failures in a fixed span of time. The addition law may be used to determine the probability for any particular compound event. For example, "at most 3" is an event occurring whether the number of failures is "exactly 0," "exactly 1," "exactly 2," or "exactly 3." Thus,

$$\begin{aligned} \Pr[\text{at most } 3] &= \Pr[0 \text{ or } 1 \text{ or } 2 \text{ or } 3] \\ &= \Pr[0] + \Pr[1] + \Pr[2] + \Pr[3] \\ &= .3679 + .3679 + .1839 + .0613 \\ &= .9810 \end{aligned}$$

Notice that the possible failure events given in Table 6–1 constitute a collectively exhaustive collection (one of these outcomes is bound to occur). The union of all the events is therefore certain and has probability 1. Notice also that the individual event probabilities in Table 6–1 sum to 1. For any collection of mutually exclusive events that are also collectively exhaustive, this must hold.

Table 6–1
Probabilities for the Number of Power Cell Failures

Number of Failures	Probability
0	.3679
1	.3679
2	.1839
3	.0613
4	.0153
5	.0031
6	.0005
7 or more	.0001
	1.0000

Application to Complementary Events

We are now ready again to consider complementary events (opposites). The union of any event A and its complement not A is certain, and by the addition law

$$\Pr[A \textit{ or } \text{not } A] = \Pr[A] + \Pr[\text{not } A] = 1$$

From this we get the following:

$$\Pr[A] = 1 - \Pr[\text{not } A]$$

Thus, if we know the probability of an event's complement, we can subtract that value from 1 to get the probability value for the desired event.

Consider again the power cell failure events in Table 6–1. Suppose we want to find the probability that the number of failures is "at least 1." "At least 1" is the same as "some," which is the opposite of "none." Thus,

$$\begin{aligned}\Pr[\text{at least } 1] &= 1 - \Pr[0] \\ &= 1 - .3679 = .6321\end{aligned}$$

Here it is simpler to subtract a single quantity from 1 than to add together the probabilities of the seven component events for "at least 1." In fact, it is often easier and faster to reach an answer indirectly by evaluating the complementary event.

General Addition Law

The preceding version of the addition law works only for *mutually exclusive* component events. We should keep in mind that mutual exclusivity is the *exception*, so events typically encountered during probability evaluations do not share this property. In the more general case, we use the following.

GENERAL ADDITION LAW

$$\Pr[A \textit{ or } B] = \Pr[A] + \Pr[B] - \Pr[A \textit{ and } B]$$

Notice that the joint probability must be subtracted from the sum of the component event probabilities. This is to avoid double accounting of events.

Example: Metal Fabrication Faults

Suppose we know that a metal fabrication process yields output with faulty bondings (FB) 10% of the time and excessive oxidation (EO) in 25% of all segments; 5% of the output has both faults. The following probabilities apply to a sample segment:

$$\Pr[FB] = .10$$

$$\Pr[EO] = .25$$

$$\Pr[FB \textit{ and } EO] = .05$$

The general addition law can be used to find the probability that a sample segment has faulty bonding, excessive oxidation, or possibly both:

$$\Pr[FB \textit{ or } EO] = \Pr[FB] + \Pr[EO] - \Pr[FB \textit{ and } EO]$$

$$= .10 + .25 - .05 = .30$$

Thus, 30% of the output has at least one of the faults.

Statistical Independence

An important relationship between two events is statistical independence:

Definition

> Two events are **statistically independent** if the probability of any one of them is unaffected by the occurrence of the other.

Independence can ordinarily be assumed from the nature of the random experiment. For instance, consider Experiment 1 involving separate tosses of two fair coins, a dime and a quarter. The events "head for the dime" H_D and "head for the quarter" H_Q may be assumed to be statistically independent.

In Experiment 2 the dime is tossed first. If a head is obtained(H_D), a fair quarter is tossed. But if a tail is obtained (T_D), a crooked two-headed quarter is tossed instead. Thus,

$$\Pr[H_Q] = \frac{1}{2} \quad \text{when } H_D \text{ occurs}$$

$$\Pr[H_Q] = 1 \quad \text{when } T_D \text{ occurs}$$

and

$$Pr[H_Q] = \frac{3}{4} \quad \text{in any repetition of the experiment}$$

(You may verify this last result using the procedures of Section 6–4.) Thus, the events H_D and H_Q for Experiment 2 are *not* statistically independent. We say that they are *dependent* events.

Statistical independence between events may be assumed in many sampling experiments, such as those sometimes encountered in quality assessment. When items are removed from a continuous production process, the probability for getting a satisfactory unit is generally assumed to be unaffected by the quality of the previous item. However, as we shall see, independence between quality events will not be the case when sampling instead from an existing production lot of fixed size.

When events are independent, a useful probability law applies.

The Multiplication Law for Independent Events

The second basic law of probability is intended for computing joint probabilities. This is achieved by multiplying together the probabilities of the component events.

MULTIPLICATION LAW FOR INDEPENDENT EVENTS

$$\Pr[A \textit{ and } B] = \Pr[A] \times \Pr[B]$$

To illustrate, suppose a perfectly balanced coin is fairly tossed twice in succession. We denote the showing faces from the first toss by H_1 and T_1 and those from the second by H_2 and T_2. The probability of two heads is the probability of the joint event H_1 *and* H_2. Since the events may be assumed to be independent, the multiplication law provides

$$\begin{aligned}\Pr[H_1 \textit{ and } H_2] &= \Pr[H_1] \times \Pr[H_2] \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\end{aligned}$$

It is easy to verify this since the random experiment involves exactly four equally likely outcomes:

$$H_1 \textit{ and } H_2 \qquad H_1 \textit{ and } T_2 \qquad T_1 \textit{ and } H_2 \qquad T_1 \textit{ and } T_2$$

and $\Pr[H_1 \textit{ and } H_2]$ must be equal to 1 divided by 4.

Of course, if we could always count and divide we wouldn't need the multiplication law. Suppose instead that the coin is lopsided, having a 25% chance of "head." Assuming that independence still applies between successive toss outcomes, the multiplication law gives

$$\begin{aligned}\Pr[H_1 \textit{ and } H_2] &= \Pr[H_1] \times \Pr[H_2] \\ &= .25 \times .25 = .0625\end{aligned}$$

which is the simplest way to logically deduce the joint probability.

***Example:* Noise-Induced Data-Transmission Errors**

Microwave telecommunication systems for data transmission are often subject to short bursts of spurious noise, which can render entire message segments useless. When identified as anomalies by parity-check codes, erroneous segments can sometimes be automatically retransmitted without human intervention. One such system stores the bits from millisecond-long segments in a buffer memory for retransmission should there be a noise interruption.

Suppose that 1% of such data segments suffer noise distortions (N), whereas the remainder are clear (C). Bursts of noise are so short that we may assume independence between C and N events for successive segments. For segment i, we have probability

.01 that there is noise (N_i) and .99 that the segment is clear (C_i). Consider just three segments, identified by 1, 2, and 3. The multiplication law for independent events allows us to evaluate the joint probability that all three segments are transmitted clearly:

$$\Pr[C_1 \text{ and } C_2 \text{ and } C_3] = \Pr[C_1] \times \Pr[C_2] \times \Pr[C_3]$$
$$= .99 \times .99 \times .99 = .9703$$

The probability that the second segment is the only one subject to noise is found by applying the multiplication law in evaluating the joint event C_1 *and* N_2 *and* C_3:

$$\Pr[C_1 \text{ and } N_2 \text{ and } C_3] = \Pr[C_1] \times \Pr[N_2] \times \Pr[C_3]$$
$$= .99 \times .01 \times .99 = .0098$$

The above joint events are also two of the elementary events in the sample space enumerating all three-segment transmission possibilities. Two more are (N_1 *and* C_2 *and* C_3) and (C_1 *and* C_2 *and* N_3), each having probability .0098. (As an exercise, you may find the remaining four elementary events and their probabilities.) To find the probability of exactly 1 noise interruption, we may apply the addition law to the respective mutually exclusive component events (the three elementary events involving exactly one N_i):

$$\Pr[\text{exactly 1 noise interruption}]$$
$$= \Pr[C_1 \text{ and } N_2 \text{ and } C_3] + \Pr[N_1 \text{ and } C_2 \text{ and } C_3] + \Pr[C_1 \text{ and } C_2 \text{ and } N_3]$$
$$= .0098 + .0098 + .0098 = .0294$$

Later in the chapter we will encounter a more general multiplication law that works even when the component events are dependent.

The above example shows that the multiplication law extends to any number of independent components. It can also be used to evaluate the probability for a joint event that may itself be the elementary event of a complex random experiment.

After using the multiplication law to establish elementary event probabilities, the *addition law* might then be applied to find the probability for a compound event. Consider the following example.

Example:* *Catastrophe and the Space Shuttle

U.S. manned space shuttle flights have proven to be extremely hazardous, with NASA officials seeing the "risk [probability] of catastrophe as roughly 1 in 145 missions for each ship."* It will take years to make major modifications that can reduce this probability, and those will not take place until the year 2005.

Assuming that one mission's success or failure is independent of the status of the others, we may use the multiplication law to find the probability that there will be one or more catastrophes in a given number of flights. This is done by first computing the probability that all flights are successful and then subtracting that probability from 1:

(1) *For two missions:*

$$\Pr[\text{at least 1 catastrophe}] = 1 - \Pr[\text{all missions succeed}]$$
$$= 1 - (144/145)(144/145)$$
$$= 1 - .986 = .014$$

(2) *For five missions:*

$$\begin{aligned}\Pr[\text{at least 1 catastrophe}] &= 1 - (144/145)^5 \\ &= 1 - .966 = .034\end{aligned}$$

(3) *For ten missions:*

$$\begin{aligned}\Pr[\text{at least 1 catastrophe}] &= 1 - (144/145)^{10} \\ &= 1 - .933 = .067\end{aligned}$$

(4) *For fifty missions:*

$$\begin{aligned}\Pr[\text{at least 1 catastrophe}] &= 1 - (144/145)^{50} \\ &= 1 - .707 = .293\end{aligned}$$

The above result is consistent with past history. There has been one catastrophe in the 49 flights through 1995.

*William J. Broad, "Risks Remain Despite NASA's Rebuilding," *New York Times*, January 28, 1996.

Problems

6–9 One Tau Beta Pi member is selected at random, and the sample space in Figure 6–3 applies. Find the following probabilities:
(a) Pr[graduate *or* CE]
(b) Pr[undergraduate *or* woman]
(c) Pr[ME *or* CE]
(d) Pr[CE *or* EE]

6–10 One Tau Beta Pi member is selected at random, and the sample space in Figure 6–3 applies. Use the count-and-divide method to find the following probabilities:
(a) Pr[graduate *and* woman]
(b) Pr[undergraduate *and* man]
(c) Pr[EE *and* man]
(d) Pr[CE *and* woman]

6–11 One Tau Beta Pi member is selected at random, and the sample space in Figure 6–3 applies.
(a) Find the probability that the selection will be an *undergraduate* assuming that (1) no other attributes are known and (2) the person's major is known to be mechanical engineering. What relationship applies between the events "undergraduate" and "mechanical engineer"?
(b) Using the count-and-divide method only, find the following:
(1) Pr[undergraduate]
(2) Pr[ME]
(3) Pr[CE]
(4) Pr[ME *and* undergraduate]
(5) Pr[CE *and* undergraduate]
(c) Use your answers to (b) above. Multiply (1) and (2). Comparing that product to (4), what do you notice? Explain.
(d) Use your answers to (b) above. Multiply (1) and (3). Comparing that product to (5), what do you notice? Explain.

6–12 Consider the data-transmission example on page 166. List all the elementary events for the transmission of three segments. Then use the multiplication law to determine the probability for each elementary event.

6–13 Using your answer to Problem 6–12 complete the following:
(a) Determine the following for the number of noise-interference events:
(1) Pr[exactly 0]
(2) Pr[exactly 1]
(3) Pr[exactly 2]
(4) Pr[exactly 3]

(b) Apply the addition law for mutually exclusive events to determine the following for the number of noise-interference events:
(1) Pr[exactly 1 *or* exactly 2] (2) Pr[at least 2] (3) Pr[at most 2]

6–14 Indicate for each of the following automobile event pairs, whether independence (1) can be confidently assumed, (2) might be true, or (3) is not expected.
(a) failure of car radio, failure of car battery
(b) transmission squeal, leaking transmission seal
(c) strong wheel vibration, high speed
(d) uneven tread wear, wheels misaligned
(e) transmission squeal, leaking fuel line

6–15 Items produced by a parts supplier have the following percentages of items satisfactory in the respective categories:

Dimension	80%
Weight	90
Texture	95

Assume an item's status is independent from category to category. Let D_S and D_U denote the events that the next item produced will be satisfactory or unsatisfactory in dimension, with W_S and W_U serving analogously to denote the events for weight and T_S and T_U for texture.
(a) Consider the sample space for the characteristics of any part. One elementary event is $D_S W_S T_S$ for getting a part satisfactory in all categories. List the sample space events.
(b) The probability of the elementary event $D_S W_S T_S$ equals Pr[D_S *and* W_S *and* T_S]. Use the multiplication law to find the probability for each elementary event listed in (a).

6–16 Problem 6–15 *continued.* A part may be classified in a variety of ways. In each case list the elementary events and then determine the probability that a part will fall in that category, using as needed the addition law.
(a) an *acceptable* part that is unsatisfactory in at most one category
(b) an *unacceptable* part that is not acceptable
(c) a *reworkable* part that is an unacceptable part that is satisfactory in dimension or satisfactory in weight
(d) a *salvageable* part that is satisfactory only in texture
(e) a part that is satisfactory at least in dimension
(f) a part that is acceptable or reworkable
(g) a part that is satisfactory at least in dimension or that is salvageable
(h) a part that is salvageable or reworkable

6–17 A production process yields 90% satisfactory items. The quality events pertaining to successive items are assumed to be independent. Find the probability of finding (1) all satisfactory, (2) none satisfactory, and (3) at least one satisfactory, when the number of selected items is (a) 2, (b) 3, (c) 4, and (d) 5.

6–18 Show that if A and B are mutually exclusive events and that neither is impossible, then they cannot be statistically independent.

6–19 For about $1 billion in new space shuttle expenditures, NASA has proposed to install new heat pumps, power heads, heat exchangers, and combustion chambers. These will lower the mission catastrophe probability to 1 in 200.*
(a) Assume that these changes are made. Then for (1) two, (2) five, (3) ten, and (4) fifty missions, calculate the probability of one or more catastrophes.
(b) Comparing your answer from (a) to the analogous probabilities on pages 167–68, do NASA's proposed changes substantially reduce the probabilities for catastrophe?

*William J. Broad, "Risks Remain Despite NASA's Rebuilding," *New York Times*, January 28, 1996.

6–3 Conditional Probability

We have now seen the important relationships between events, the main forms of compound events, and several ways to compute probabilities for a variety of outcome types. This section consolidates that knowledge and establishes a conceptual framework for further extensions of probability theory.

Consider how the probability value of one event is affected by the occurrence of another. It is more likely to rain tomorrow if there is a solid cloud cover than if there are patches of blue. Electronic equipment is more likely to fail in high temperatures than in moderate ones—and we might be uncertain which operating environment will apply. Items produced in antiquated plants will generally involve more rejects than those created in modern facilities (although we might not know where a particular item is made).

A probability value computed under the assumption that another event is going to occur is a **conditional probability**. For the events "rain" and "solid overcast," the following value might apply:

$$\Pr[\text{rain} \mid \text{solid overcast}] = .90$$

The event "rain" is listed first, and .90 is the conditional probability of that event; the condition—given event—"solid overcast," is listed second, and the vertical bar is a shorthand representation for "given that." A different rain probability applies given some other event:

$$\Pr[\text{rain} \mid \text{blue patches}] = .30$$

Should no stipulations be made regarding sky conditions, we might have

$$\Pr[\text{rain}] = .20$$

which is an **unconditional probability**.

It may instead be possible to compute a conditional probability directly from known probability values using the following:

CONDITIONAL PROBABILITY IDENTITY

$$\Pr[A \mid B] = \frac{\Pr[A \text{ and } B]}{\Pr[B]}$$

Consider again the events "undergraduate" and "electrical engineer" for the student lottery in Figure 6–3. We have

$$\Pr[UG \mid EE] = \frac{\Pr[UG \text{ and } EE]}{\Pr[EE]} = \frac{\frac{5}{38}}{\frac{12}{38}} = \frac{5}{12}$$

which expresses the probability that the selected student is an undergraduate given that he or she majors in electrical engineering. Notice that the numerator is the joint probability and the denominator is the given event's probability. A total reversal of the given and uncertain events is achieved by switching divisors:

$$\Pr[EE \mid UG] = \frac{\Pr[UG \text{ and } EE]}{\Pr[UG]} = \frac{\frac{5}{38}}{\frac{19}{38}} = \frac{5}{19}$$

Of course, the same results could have been achieved using the basic definition: With 5 out of 12 electrical engineering majors being undergraduates, we have

$\Pr[UG \mid EE] = \frac{5}{12}$. Likewise, $\Pr[EE \mid UG] = \frac{5}{19}$, since there are 5 electrical engineers out of 19 undergraduate students.

Although the foregoing identity is always true, it may not always be suitable for computing a conditional probability. We must know the values Pr[*A and B*] and Pr[*B*] in order to use it to evaluate $\Pr[A \mid B]$.

Example: Crooked and Fair Dice

A box contains four crooked (*C*) pairs of dice. One member of each pair has 3 dots on every face and the other has 4 dots on all sides. The box also contains six fair (*F*) pairs of dice. The faces of each of these dice have a different number of dots from 1 through 6.

One pair is selected at random and rolled. Given that the pair is fair, what is the conditional probability that a 7-sum results when the number of dots on the two up sides are added together?

The answer is

$$\Pr[\text{7-sum} \mid F] = \frac{6}{36} = \frac{1}{6}$$

This is found by *counting and dividing* the respective numbers of possibilities. (You may verify that there are 36 equally likely possibilities, six of which result in a sum of 7.)

The conditional probability identity cannot be used since it requires

$$\Pr[\text{7-sum } and\ F]$$

in the numerator, and the value of this is not presently known.

Establishing Independence by Comparing Probabilities

Recall that two events are independent if the probability of one is unaffected by the occurrence of the other. We may apply conditional probability concepts to more precisely define independence.

Definition

Two events *A* and *B* are **statistically independent** whenever $\Pr[A|B] = \Pr[A]$ or $\Pr[B|A] = \Pr[B]$.

Stated less formally, *A* and *B* are independent if one event has unconditional probability equal to its own conditional probability given the other event.

We may use this fact to establish statistical independence or dependence by comparing probability values. (But keep in mind that the independence may be simply assumed to hold between some outcomes—such as the showing faces from two coin tosses—wherein the nature of the random experiments makes the relationship obvious.)

Returning to the student lottery in Figure 6–1, we can see that *W* (woman) and *G* (graduate student) are *dependent* because

$$\Pr[W \mid G] = \frac{8}{19} \quad \text{does not equal} \quad \Pr[W] = \frac{17}{38}$$

This is because the conditional probability of *W* given *G* differs from the unconditional probability of *W*. (You may also establish that $\Pr[G \mid W] \neq \Pr[G]$). The same relationship applies here for all sex and student-level event pairs.

Problems

6–20 Consider Experiment 2 on page 165. Find the following.
(a) (1) $\Pr[H_Q \mid H_D]$ (2) $\Pr[T_Q \mid H_D]$
(b) (1) $\Pr[H_Q \mid T_D]$ (2) $\Pr[T_Q \mid T_D]$

6–21 A floppy disk is randomly selected from those listed in Figure 6–2. Find the following for its quality state.
(a) (1) Pr[S] (2) Pr[S | workable (S, U, I, or O)]
(b) (1) Pr[B] (2) Pr[B | unworkable (F or B)]
(c) (1) Pr[ready to use (S, I, or O)] (2) Pr[ready to use | workable (S, U, I, or O)]
(d) (1) Pr[needs formatting (U or O)] (2) Pr[needs formatting | workable (S, U, I, or O)]

6–22 Refer to Problem 6–21 and your answers to that exercise. For each of the following event pairs indicate whether the events are (1) independent or (2) dependent.
(a) S workable
(b) B unworkable
(c) ready to use workable
(d) needs formatting workable

6–23 Consider a random experiment involving three boxes, each containing a mixture of red (R) and green (G) balls, with the following quantities:

A	B	C
5 R	14 R	8 R
5 G	6 G	12 G

The first ball will be selected at random from box A. If that ball is red, the second ball will be drawn from box B; otherwise, the second ball will be taken from box C.

Let R_1 and G_1 represent the color of the first ball, R_2 and G_2 the color of the second. Determine the following probabilities. (*Hint:* The conditional probability identity will not work.)

(a) $\Pr[R_1]$ (c) $\Pr[R_2 \mid R_1]$ (e) $\Pr[G_2 \mid G_1]$
(b) $\Pr[G_1]$ (d) $\Pr[R_2 \mid G_1]$ (f) $\Pr[G_2 \mid R_1]$

6–24 Consider the student lottery summarized in Figure 6–3. Find the following conditional probabilities:
(a) $\Pr[EE \mid G]$ (b) $\Pr[ME \mid G]$ (c) $\Pr[G \mid EE]$ (d) $\Pr[G \mid ME]$

6–25 Refer to Problem 6–24. Find the following conditional probabilities:
(a) $\Pr[ME \mid U]$ (b) $\Pr[CE \mid U]$ (c) $\Pr[U \mid ME]$ (d) $\Pr[U \mid CE]$

6–4 The Multiplication Law, Probability Trees, and Sampling

With basic probability concepts and tools firmly in hand, we set the stage for deductive statistics by considering the sampling process. We can now find probabilities for particular sample outcomes. Later in the text this background will prove useful in developing the concepts of inductive or inferential statistics.

In this section we introduce two new tools while laying the above groundwork. A *general* multiplication law is given that works whether or not component events are independent. To help explain this procedure, we introduce a useful display, the probability tree diagram, which conveniently organizes information about random experiments that involve a series of uncertainties.

The General Multiplication Law

The multiplication law introduced earlier requires that the component events be independent. But independence is often the exception. This is especially true in testing and evaluation—in which the desired event and the experimental result should be dependent. For example, a seismic survey result (an uncertainty) should be statistically dependent on the presence of oil. (Otherwise seismic predictions would be worthless in finding drilling locations.)

We have a general version of the multiplication law.

GENERAL MULTIPLICATION LAW

$$\Pr[A\ and\ B] = \Pr[A] \times \Pr[B \mid A]$$

and

$$\Pr[A\ and\ B] = \Pr[B] \times \Pr[A \mid B]$$

Notice that the probability for the first event in the product is an unconditional one, whereas the second is a conditional probability for the other event, given the first. The multiplication law given earlier is a special case of this because when A and B are independent, $\Pr[A \mid B] = \Pr[A]$ and $\Pr[B \mid A] = \Pr[B]$.

Illustration: Quality-Control Inspector's Probabilities

To illustrate, suppose that a quality-control inspector has established that 10% of all electric assemblies fail (F) the circuit test. Twenty percent of the failing items have poor (P) solder joints. This experience provides the following probabilities:

$$\Pr[F] = .10$$
$$\Pr[P \mid F] = .20$$

The multiplication law gives the joint probability that a particular assembly will fail and have poor solder joints:

$$\begin{aligned}\Pr[F and P] &= \Pr[F] \times \Pr[P \mid F] \\ &= .10 \times .20 \\ &= .02\end{aligned}$$

Altogether, 2% of all assemblies will have both deficiencies.

Illustration: Oil Wildcatter's Probabilities

A second illustration is provided by the experience of an oil wildcatter. He has established that there is a 40% chance for oil (O) beneath a particular site. Furthermore, from past experience with seismic testing, the procedure is known to be 80% reliable. That is, for known oil-bearing sites there is an 80% probability of getting favorable (F) seismic results, and when the site is known to be dry, there is a 90% probability of getting an unfavorable (U) prediction. Thus,

$$\Pr[O] = .4$$
$$\Pr[\text{not } O] = 1 - .4 = .6$$
$$\Pr[F \mid O] = .8$$
$$\Pr[U \mid \text{not } O] = .9$$

The general multiplication law may be used to compute joint probabilities for geological and seismic events:

$$\Pr[O \textit{ and } F] = \Pr[O] \times \Pr[F \mid O] = .4 \times .8 = .32$$

$$\Pr[\text{not } O \textit{ and } U] = \Pr[\text{not } O] \times \Pr[U \mid \text{not } O] = .6 \times .9 = .54$$

Example:* *Using Probability to Help Identify Space Shuttle Improvements

One adviser has offered NASA some assistance in helping to improve the safety of the U.S. manned space shuttle flights, for which she has read an article reporting the probability of any single mission meeting with catastrophe as roughly 1 in 145. Given that a mission catastrophe occurs, the following conditional probabilities apply for its major cause.

Major Cause	Conditional Probability
(1) E_1: Faulty ignition or booster separation	.60
(2) E_2: Main engine cutoff or external tank separation	.23
(3) E_3: Auxiliary power shutdown	.01
(4) E_4: De-orbit accident	.02
(5) E_5: Reentry failure	.08
(6) E_6: Final flight accident	.02
(7) E_7: Landing accident	.04
	1.00

Source: William J. Broad, "Risks Remain Despite NASA's Rebuilding," *New York Times*, January 28, 1996.

Applying the multiplication law, the following joint probability applies:

$$\Pr[E_1 \textit{ and } \text{Catastrophe}] = \Pr[E_1 \mid \text{Catastrophe}] \times \Pr[\text{Catastrophe}] = .60(1/145) = .00414$$

Thus, there is about a .4% chance that any particular mission will fail because of faulty ignition or booster separation.

The adviser suggests that NASA make a list of possible shuttle system improvements and the corresponding reductions in the conditional probabilities listed in the above table. That might help establish budget priorities.

Multiplication Law for Several Events

The general multiplication law can be extended to several components. Consider events $A_1, A_2, \ldots, A_n$. We have

$$\Pr[A_1 \text{ and } A_2 \text{ and } A_3 \ldots \text{ and } A_n] = \Pr[A_1] \times \Pr[A_2 \mid A_1] \times \Pr[A_3 \mid A_1 \text{ and } A_2] \times \ldots \times \Pr[A_n \mid A_1 \text{ and } A_2 \text{ and } A_3 \ldots \text{ and } A_{n-1}]$$

The Probability Tree Diagram

A convenient summary display is the **probability tree**. Figure 6–4 shows the probability tree diagram for the oil wildcatting illustration. Each event is represented as a **branch** in one or more **event forks**. A probability tree can be especially convenient for random experiments having events that occur at different times or stages.

In probability trees, time ordinarily moves from left to right. Since the geology events precede the seismic ones, the branches for oil (O) and not oil (not O) appear in the leftmost event fork. Each of these is followed by a separate fork representing the seismic events, with one branch for favorable (F) and one for unfavorable (U). The complete tree then exhibits each final outcome as a single *path* from beginning to end. The oil wildcatter's tree has four paths: oil-favorable, oil-unfavorable, not-oil-favorable, and not-oil-unfavorable. Each of these corresponds to a distinct joint event.

The probabilities for each event are placed alongside its branch. The values listed in the left fork—the geology events fork—are $\Pr[O] = .4$ and $\Pr[\text{not } O] = .6$. Since that event fork has no predecessor, these probabilities are unconditional ones.

Probabilities for events at later stages will all be conditional probabilities, with the branch or subpath leading to the branching point signifying the given events. The top

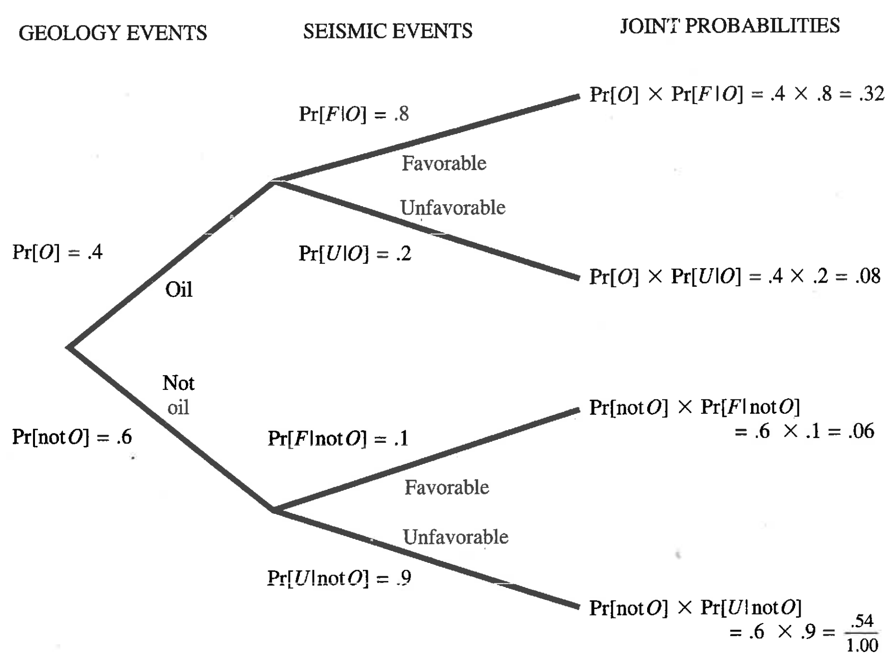


Figure 6–4 *Probability tree diagram for oil wildcatter.*

seismic events fork lists the probabilities $\Pr[F \mid O] = .8$ for favorable and $\Pr[U \mid O] = .2$ for unfavorable; since that fork is preceded by the oil (O) branch, both conditional probabilities involve O as the given event. A completely different set of conditional probabilities, $\Pr[F \mid \text{not } O] = .1$ and $\Pr[U \mid \text{not } O] = .9$, apply in the bottom event fork; that branching point is preceded by the not oil (not O) branch, so that not O is the given event.

The events emanating from a single branching point are mutually exclusive and collectively exhaustive, so that exactly one of them must occur. All the probabilities on branches within the same fork must therefore sum to 1.

The probability tree is very convenient for computing joint probabilities. Since each joint event is represented by a path through the tree, its probability is found by multiplying together all the individual branch probabilities along its path. For instance, the topmost path represents the outcome sequence oil-favorable. The corresponding joint probability is

$$\begin{aligned}\Pr[O \textit{ and } F] &= \Pr[O] \times \Pr[F \mid O] \\ &= .4(.8) \\ &= .32\end{aligned}$$

All of the oil wildcatter's joint probabilities are listed in Figure 6–4 at the terminus of the respective event path. Notice again that, because the joint events themselves are mutually exclusive and collectively exhaustive, they sum to 1.

Probability and Sampling

Probability is especially important in summarizing potential sampling results. Because multiple observations are usually involved in statistical evaluations, the probability calculations can be complicated. Probability trees can be helpful in organizing these computations.

Illustration: *Quality-Control Sampling with Memory Chips*

To illustrate, consider a shipment of 100 memory chips received by a computer manufacturer. Each chip is either good (G) or defective (D). The decision to accept or reject the shipment will be based on a sample of three chips selected at random. Although the inspector cannot know ahead of time how many defectives there are, let's consider a shipment having exactly 90 good chips and 10 defective ones.

Figure 6–5 shows a probability tree that summarizes the essential information. The first observation is represented by the two branches in the leftmost event fork. It is convenient to use the abbreviation G_1 to denote the event that "the first chip is good" and D_1 that "the first chip is bad." The subscripts help distinguish the results from different observations. Each branch for the first observation leads to a separate event fork for the second item, with both forks having a G_2 and a D_2 branch. Together, the initial observations create four distinct circumstances under which the third observation might occur, and each of these is represented by a separate fork having a G_3 and a D_3 branch.

The probabilities for each branch are found using the count-and-divide method, according to the number of remaining good and defective chips up to that point in the tree. The initial fork has probabilities

$$\Pr[G_1] = \frac{90}{100} \qquad \text{and} \qquad \Pr[D_1] = \frac{10}{100}$$

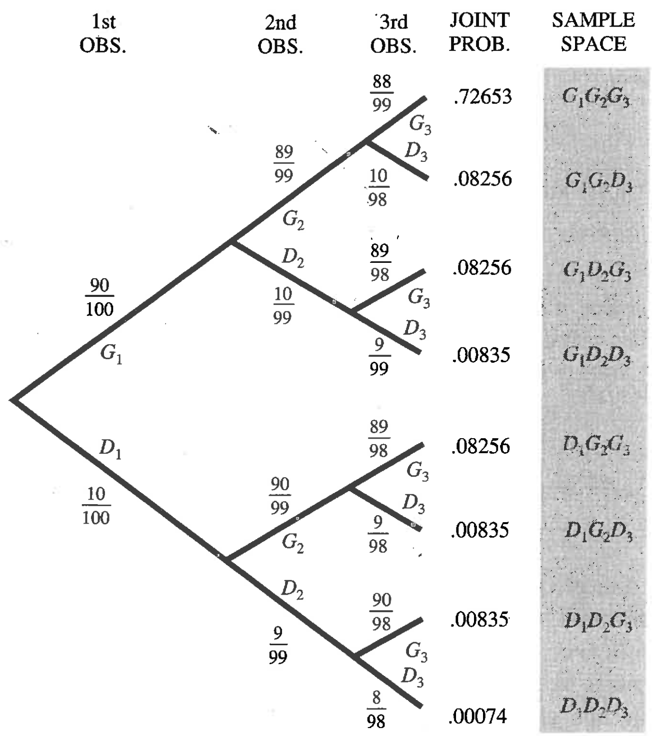


Figure 6–5 *Probability tree diagram for a shipment of items sampled without replacement.*

Because the quality mix of the remaining items varies at the second stage, the probabilities for G_2 and D_2 differ in the two event forks for that observation. The shown values are *conditional probabilities*. The top set applies when G_1 has occurred, when there are only 89 remaining good chips and 10 defective. With 99 chips then available for testing, the count-and-divide procedure provides

$$\Pr[G_2 \mid G_1] = \frac{89}{99} \qquad \text{and} \qquad \Pr[D_2 \mid G_1] = \frac{10}{99}$$

Likewise, the lower second-stage fork involves

$$\Pr[G_2 \mid D_1] = \frac{90}{99} \qquad \text{and} \qquad \Pr[D_2 \mid D_1] = \frac{9}{99}$$

since at that point—with one defective chip removed—the tree shows that 90 good chips and 9 defectives remain.

The third-stage forks involve probabilities reflecting the respective histories of good and defective items. The fork at the top right is preceded by G_1 and G_2 branches, so the conditional probabilities are

$$\Pr[G_3 \mid G_1 \textit{ and } G_2] = \frac{88}{98} \qquad \text{and} \qquad \Pr[D_3 \mid G_1 \textit{ and } G_2] = \frac{10}{98}$$

It is easy to verify the remaining third-stage probabilities.

Every path through the tree in Figure 6–5 corresponds to a distinct final outcome, each of which is summarized in the sample space listed in the box beside the tree. The probability for each of these elementary events is found by multiplying together the branch probabilities on its respective path. For instance, to get the second elementary event probability, we multiply as follows:

$$\Pr[G_1 G_2 D_3] = \frac{90}{100} \times \frac{89}{99} \times \frac{10}{98} = .08256$$

In doing this, we are actually applying the multiplication law for several joint events:

$$\Pr[G_1 \text{ and } G_2 \text{ and } D_3] = \Pr[G_1] \times \Pr[G_2 \mid G_1] \times \Pr[D_3 \mid G_1 \text{ and } G_2]$$

The inspector can summarize the information contained in the probability tree in terms of the number of defectives in the shipment, as shown in Table 6–2. Using the values from the last column, we can determine the probability that a particular shipment would contain more than 1 defective:

$$\begin{aligned}\Pr[\text{more than 1 defective}] &= .02505 + .00074 \\ &= .02579\end{aligned}$$

If she rejects shipments having more than 1 defective item, less than three percent of shipments similar to the above would be unacceptable.

Independent Sample Observations: The preceding illustration involves **sampling without replacement**, since inspected sample units are set aside. Although it seems inherently wasteful to do so (and even impossible when testing destroys the items), some inspection schemes would put the inspected items back into the original population, allowing them an equal chance with the rest to be chosen for each subsequent observation. Those plans involve **sampling with replacement**. As we shall see, sampling with replacement simplifies the probability calculations.

Figure 6–6 shows the probability tree diagram that would apply if the inspector sampled with replacement. Notice that all the G branches have identical probabilities of

Table 6–2
Probabilities for the Number of Defective Memory Chips When Three Items Are Inspected in a Random Sample without Replacement

Number of Defectives	Corresponding Elementary Events	Elementary Event Probability	Defectives Probability
0	$G_1G_2G_3$	.72653	.72653
1	$D_1G_2G_3$	.08256	.24768
	$G_1D_2G_3$	.08256	
	$G_1G_2D_3$	.08256	
		.24768	
2	$D_1D_2G_3$	.00835	.02505
	$D_1G_2D_3$	.00835	
	$G_1D_2D_3$	.00835	
		.02505	
3	$D_1D_2D_3$	.00074	.00074
			1.00000

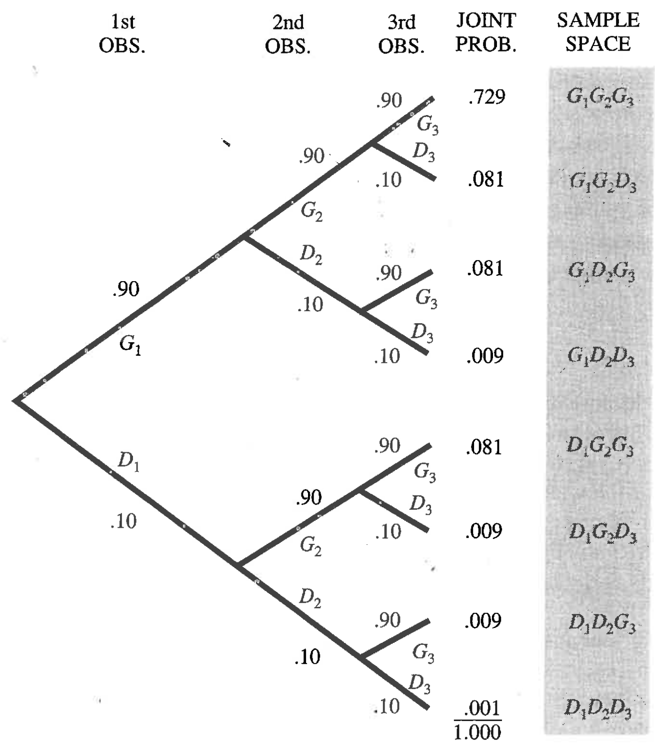


Figure 6–6
Probability tree diagram for items from a production process sampled with replacement.

.90, and similarly the D probabilities are all .10. Since sampled items are put back, the probabilities for later quality events are unaffected by what happens earlier. In effect, successive quality events are *statistically independent*. Independence between successive sample observations arises naturally when sampling from a continuing production line, when replacement or nonreplacement would yield identical probabilities due to the theoretically infinite population size.

The probabilities for the number of defectives found under sampling with replacement are given in Table 6–3. The probabilities are obtained in the same manner as before. To help in comparing the two procedures, the two sets of probabilities are given side by side.

Table 6–3
Probabilities for the Number of Defective Memory Chips When Three Sample Items Are Inspected

	Probabilities for Sampling	
Number of Defectives	**Without Replacement**	**With Replacement**
0	.72653	.729
1	.24768	.243
2	.02505	.027
3	.00074	.001
	1.00000	1.000

Notice how close the two sets of values are. Since the probabilities computed for sampling without replacement can involve some cumbersome calculations, they are often *approximated* by the more cleanly computed values that strictly apply only when there is replacement.

The sets of probability information in Table 6–3 are examples of *probability distributions*, which are described in detail in Chapter 7. When sampling is done with replacement—or more generally, when successive observations are independent—the *binomial distribution* applies. Under sampling without replacement from finite populations the *hypergeometric distribution* applies.

Problems

6–26 An oil wildcatter has assigned a .30 probability for striking gas (G) under a particular leasehold. He has ordered a seismic survey that has a 90% positive reliability (given gas it confirms gas [C] with a probability .90), but only 70% negative reliability (given no gas it denies gas [D] with probability .70).

(a) Establish the following probabilities:
(1) Pr[G *and* C] (2) Pr[no G *and* C] (3) Pr[no G *and* D] (4) Pr[G *and* D]

(b) Apply the addition law to your answer to (a) to find
(1) Pr[C] (2) Pr[D]

(c) Using your answers to (a) and (b), find the following probabilities:
(1) Pr[$G|C$] (2) Pr[no $G|D$]

(d) Construct a probability tree with gas events in the first stage and seismic events in the second.

6–27 A project manager is creating the design for a new engine. He judges that there will be a 50–50 chance that it will have high-energy (H) consumption instead of low (L). Historically, 30% of all high-energy engines have been approved (A) with the rest disapproved (D), while 60% of all low-energy engines have been approved.

(a) Construct a probability tree for the project manager's engine events.

(b) What is the probability that his design will result in an approved engine?

6–28 Repeat Problem 6–26 (a)-(c) for a new site where gas has a .20 probability, the positive reliability is 80%, and the negative reliability is 90%.

6–29 A quality-control inspector is testing sample output from a production process for widgets wherein 95% of the items are satisfactory (S) and 5% are unsatisfactory (U). Three widgets are chosen randomly for inspection. The successive quality events may be assumed independent.

(a) Construct a probability tree diagram for this experiment, identify all the sample-outcome elementary events, and compute the respective joint probabilities.

(b) Find the probabilities for the following numbers of unsatisfactory items:
(1) none (3) exactly 2 (5) at least 1
(2) exactly 1 (4) exactly 3 (6) at most 2

6–30 A user of the widgets receives a shipment of 5 dozen containing 95% satisfactory. A sample of 3 widgets is selected *without* replacement. Repeat (a) and (b) from Problem 6–29.

6–31 Two chapters of a civil engineering society each have 60% licensed members (the rest are unlicensed), with some on public payrolls (and the rest privately employed). A questionnaire is being sent to a random sample of the members. Consider the characteristics of any one of the members surveyed.

(a) Chapter A has just as many licensed engineers on public payrolls as on private ones. This makes any professional status event *independent* of any employment sector event. Construct a joint probability table having one row for each license category and one column for each sector by first finding the marginal probabilities and then using the multiplication law to obtain the joint probabilities, which in this case may be expressed as the products of the two respective marginal probabilities.

(b) In Chapter B only 40% of the licensed engineers are public employees. It follows that any professional status event is *dependent* on any employment sector event. This means that no joint probability is equal to the product of the respective marginal probabilities. Construct the joint probability table.

6–32 Suppose that major system changes may be made to reduce the single-mission cause-of-failure probabilities for the space shuttle. The respective percentage reductions in the joint probability for mission failure and the costs of the changes are listed below.

Major Cause i	Percentage Reduction in Pr[E_i *and* Catastrophe]	Cost of Change ($ billion)
(1) E_1: Faulty ignition or booster separation	50%	2.5
(2) E_2: Main engine cutoff or external tank separation	40	.5
(3) E_3: Auxiliary power shutdown	50	.2
(4) E_4: De-orbit accident	50	.5
(5) E_5: Reentry failure	30	.7
(6) E_6: Final flight accident	50	.2
(7) E_7: Landing accident	25	.1

(a) Using the data on page 174, compute the original joint probabilities for the major cause and catastrophe events.

(b) Assume that only one of the changes listed above is made. Reduce, one change at a time, the respective joint probabilities computed in (a). Then compute the new overall mission catastrophe probability by summing the original joint probabilities with the modified one.

(c) Which change is most cost effective? Which change is least cost effective?

6–5 Predicting the Reliability of Systems

An important application of probability is reliability analysis used to predict an overall system's reliability, using as building blocks the reliabilities of individual components. It is convenient to define reliability in terms of a survival probability, using the notation

$$R_i(t) = \Pr[\text{component } i \text{ survives beyond time } t]$$

The reliability of the system is then a function of the reliabilities of its components:

$$R_s(t) = f\{R_1(t), R_2(t), \ldots\}$$

When t is fixed at a specified level, a closed form may be obtained for $R_s(t)$ that coincides with the system logic. There are two primary system forms: one applies when the components are arranged in *parallel* and another when they are joined in a *series*. More complex cases involve modular subsystems as building blocks. We can portray the logic of systems schematically in similar fashion to electrical circuits.

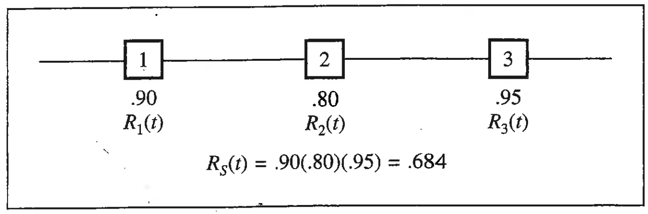


Figure 6–7
Logic for system with series components.

Systems with Series Components

A **series system** is one that performs satisfactorily as long as *all* components are fully functional. Figure 6–7 shows the logic of a small system. The system is analogous to an electrical circuit that works as long as current flows through all components. Failure of any component will cause the entire system to fail. Propulsion on the NASA space shuttle is an example of a series system since failure on any of the booster rockets will result in an aborted mission.

Under the series logic, the system will survive past time t only if its components do so. System reliability, therefore, is equal to the product of the component reliabilities:

$$R_S(t) = R_1(t)R_2(t) \ldots R_n(t)$$

The above follows directly from the multiplication law for independent events.

The system in Figure 6–7 involves three components with reliabilities $R_1(t) = .90, R_2(t) = .80$, and $R_3(t) = .95$. The system reliability is

$$R_S(t) = .90(.80)(.95) = .684$$

Systems with Parallel Components

A **parallel system** is one that performs as long as *any one* of its components remains operational. Figure 6–8 is a schematic for a simple parallel system. The electrical analogy is a circuit through which current flows as long as any component works, providing an open path. All components must fail before the parallel system does. For example, batteries for a handheld calculator might be arranged in parallel, so that the calculator may function as long as at least one battery provides sufficient power.

A parallel system will survive past time t if at least one component survives. Failure of the system occurs only if all components fail. The probability of system failure is therefore the product of those failure probabilities (each being 1 minus the respective survival probability):

$$\Pr[\text{system failure at or before } t] = [1 - R_1(t)][1 - R_2(t)] \ldots [1 - R_n(t)]$$

Thus, system reliability must be the complement to this, as follows:

$$R_S(t) = 1 - [1 - R_1(t)][1 - R_2(t)] \ldots [1 - R_n(t)]$$

The above follows from the multiplication law for independent events and the property for computing complementary event probabilities.

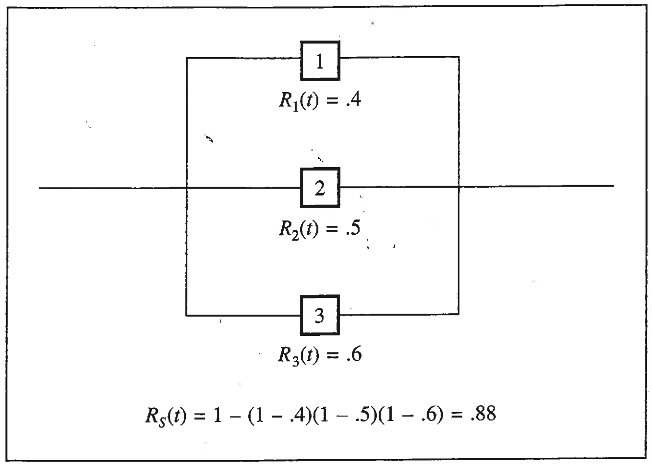


Figure 6–8
Logic for system with parallel components.

The parallel system shown in Figure 6–8 has component reliabilities of $R_1(t) = .4$, $R_2(t) = .5$, and $R_3(t) = .6$. That system's reliability is

$$R_S(t) = 1 - (1 - .4)(1 - .5)(1 - .6) = 1 - .12 = .88$$

Increasing System Reliability

A design issue in systems is how to increase system reliability. One way this might be accomplished is by raising the reliability of individual components (perhaps by substituting better materials, such as gold wiring instead of copper). To illustrate, consider the system in Figure 6–9(a). Suppose that the reliability of component 2 can be raised to .95, as in Figure 6–9(b), by substituting a more expensive item.

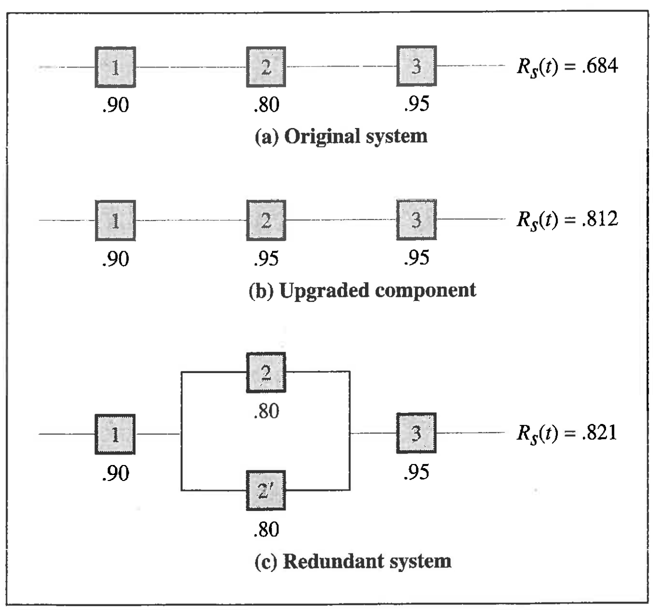
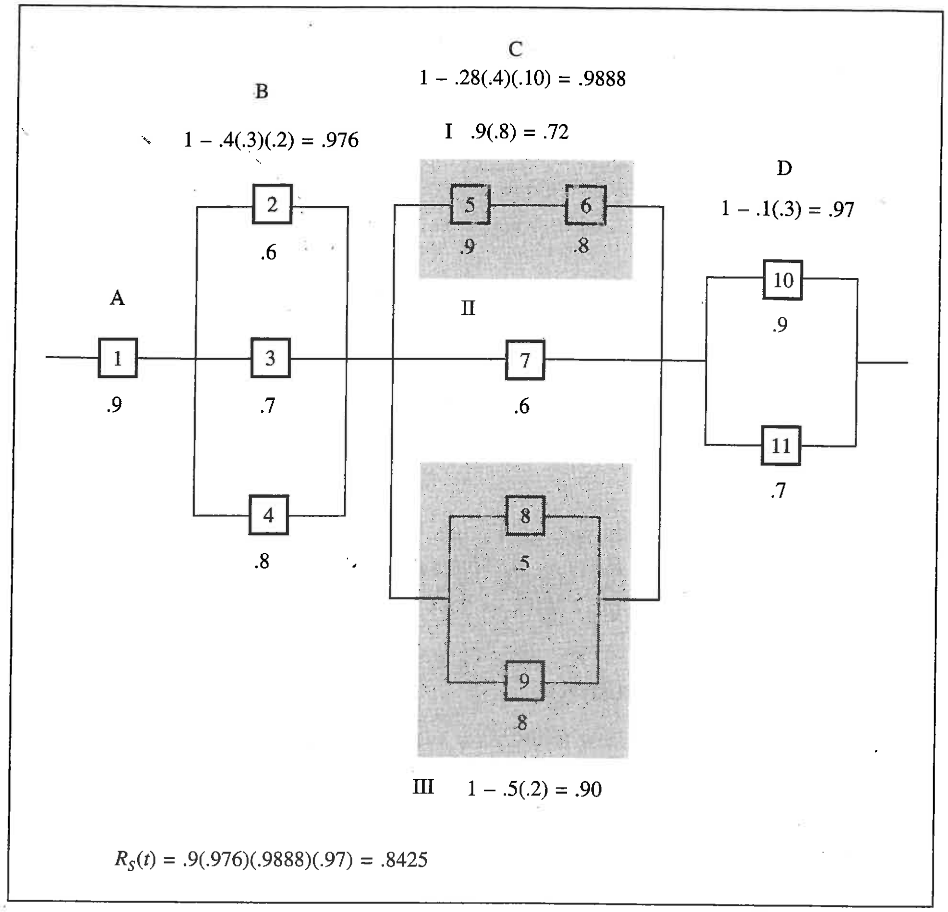


Figure 6–9
Two approaches for increasing system reliability.

Figure 6–10
Modular system with series systems.



The system reliability then becomes

$$R_S(t) = .90(.95)(.95) = .812$$

which is a substantial boost over the original level of .684.

A second approach to boosting system reliability is to duplicate certain components, replacing individual items with a parallel subsystem composed of two or more identical units. In effect, reliability is increased through system *redundancy*. Consider the system again, but now modified as in Figure 6–12(c). In this system, component 2 has an identical partner 2′, joined in parallel. The new modular subsystem has reliability

$$1 - [1 - R_2(t)][1 - R_{2'}(t)] = 1 - (1 - .8)(1 - .8) = .96$$

and the overall system reliability becomes

$$R_S(t) = .90(.96)(.95) = .821$$

Complex Modular Systems

Individual components may be aggregated into subsystems according to their interrelationships. Figure 6–10 shows a modular system with four primary subsystems—A, B,

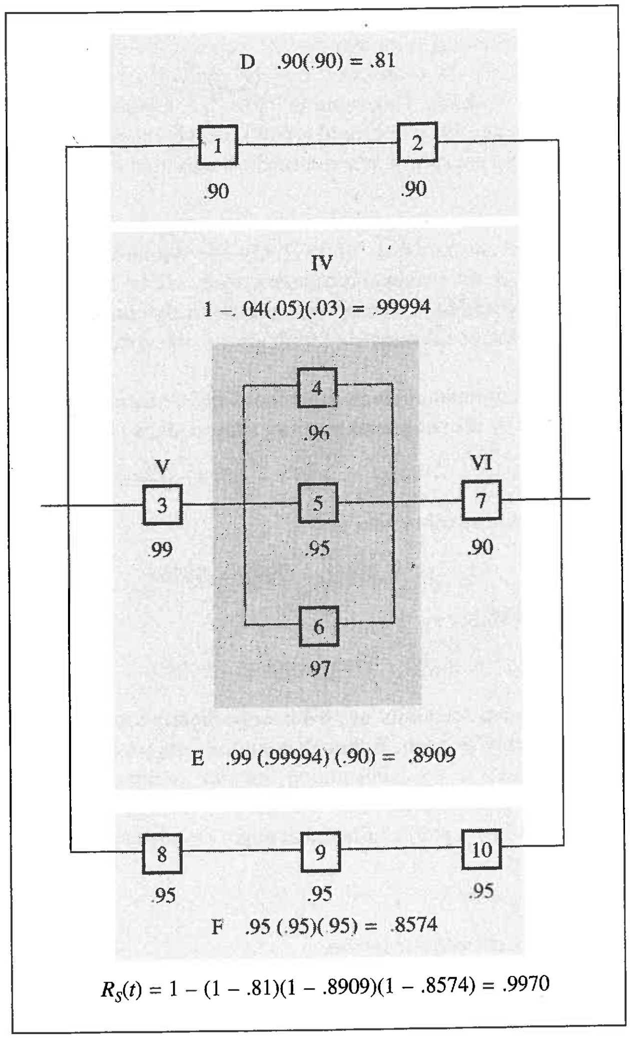


Figure 6–11 *Modular system with parallel subsystems.*

C, and D—arranged in series. To find the system reliability, we must first establish separate reliabilities for each subsystem. The most complex module is C, with three parallel components or subsystems of its own—I, II, and III. Before we can get the reliability of C, these must each be evaluated.

Figure 6–11 shows another module with three subsystems—D, E, and F—arranged in parallel. Each primary subsystem involves a series structure, and their reliabilities must be established before we can compute overall system reliability. Subsystem E has three components or subsystems, including IV, itself a parallel subsystem.

Example: *Reliability of a Personal Computer System*

A software engineer's personal computer system is shown in Figure 6–12. The reliabilities given there represent the probability that the component performs satisfactorily throughout a specific workday. This setup involves 5 components: the power supply (reliability .995), PC unit (.999), keyboard (.9999), peripheral memory subsystem, and printer subsystem. These are shown in *series* since all must function in order for the system to be usable.

The peripheral memory subsystem includes two floppy disk drives (A and B, both with .99 reliability) and one hard disk drive C (.95). The system logic shows those components in *parallel*, since the personal computer system will be functional as long as at least one of them is operational. The printer subsystem also has two parallel components, one laser (.99) and one dot-matrix (.999) printer; the system is usable as long as one of these works.

The two subsystem reliabilities must be found before the overall system reliability can be computed. For the peripheral memory subsystem, we have the reliability

$$1 - (1 - .99)(1 - .99)(1 - .95) = .999995$$

The reliability of the printer subsystem is

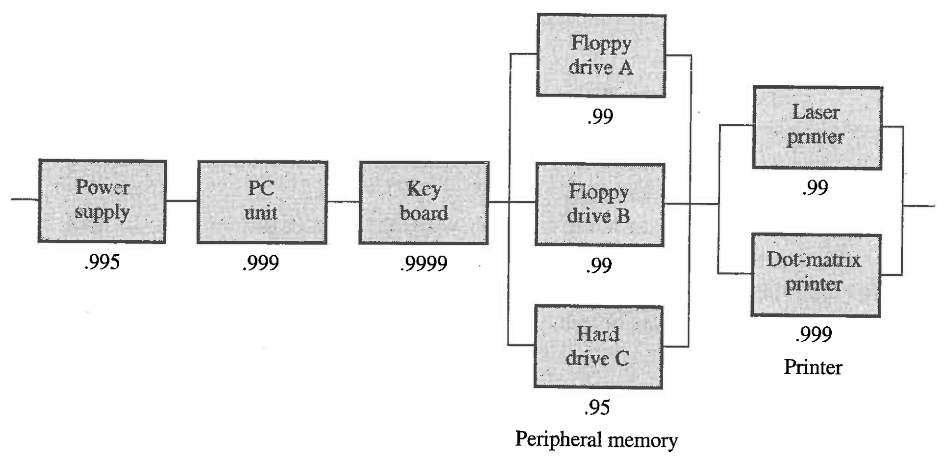
$$1 - (1 - .99)(1 - .999) = .99999$$

The overall system reliability is

$$R_S = .995(.999)(.9999)(.999995)(.99999) = .994$$

Notice that a system reliability of .994 is only slightly lower than the .995 reliability of the power supply by itself. Although minuscule improvements in system reliability might be achieved by substituting hardier components, no significant improvement in survival probability can be achieved without greater power reliability. Since power is supplied by a public utility, that might be achieved only by installing an auxiliary power system.

Figure 6–12 *Logic for personal computer system.*



Problems

6–33 Consider a system having four components with reliabilities through time t of (1) .9, (2) .7, (3) .8, and (4) .6. Find the system reliability assuming (a) series logic and (b) parallel logic.

6–34 Which has a greater reliability: (a) a series system with 4 components, each having .99 reliability, or (b) a parallel system with 4 components, each having .90 reliability?

6–35 Consider a sound system with logic and single-hour listening reliability as given in Figure 6–13. Find the system reliability.

6–36 Consider the system in Figure 6–14. The reliabilities apply to the first 100 hours of operation.

(a) Determine the system reliability.
(b) Upgrade component 4 to .9 reliability. Recompute the system reliability.
(c) A redundant system may be created instead by substituting two type-4 items, each at the original reliability. Recompute the reliability for that system.

6–37 Refer to the sound system in Figure 6–13. Suppose that all components have independent exponential failure-time distributions, with the following mean failure rates:

(1) Power: .0001/hour
(2) Amplifier: .0002
(3) CD player: .0050
(4) Tuner: .0010
(5) Tape deck: .0020
(6) Turntable: .0090
(7,8) Each speaker: .0010

Find the system reliability for the following number of hours of play: (a) 5, (b) 10, and (c) 100. [Hint: Use $R_i(t) = e^{-\lambda t}$, where λ is the mean failure rate.]

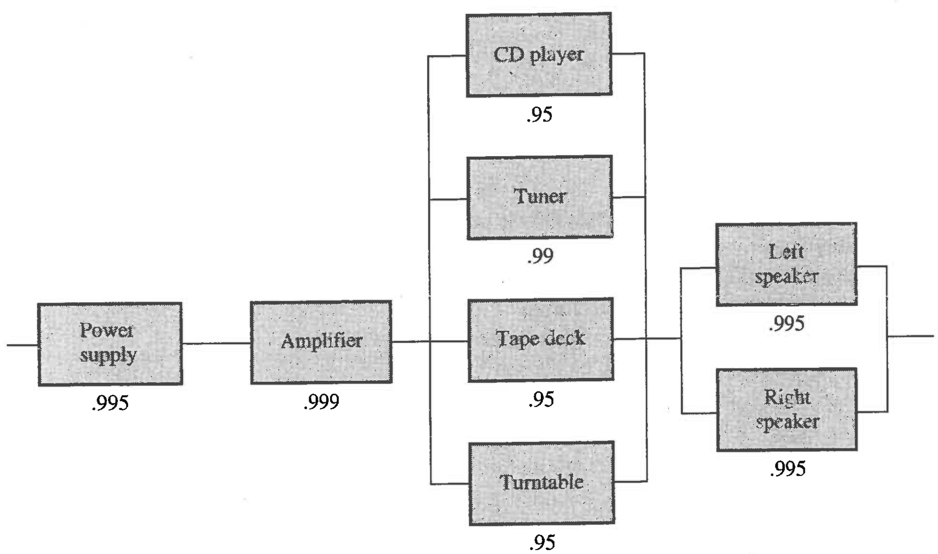
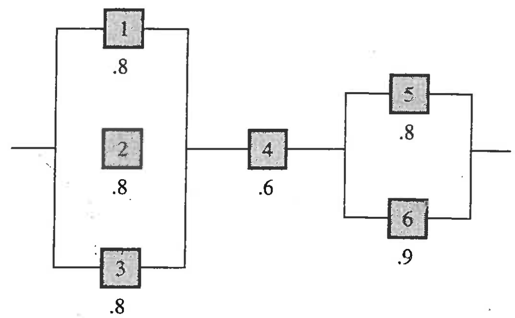


Figure 6–13 *Logic for sound system.*

Figure 6–14



6–38 Consider the system in Figure 6–14. For each percentage point increase in system reliability there will be a $10 payoff. The following possibilities apply:
(a) Component 4 can be upgraded for $100 to reliability .9.
(b) Component 4 can be duplicated any number of times in a redundant system at a cost of $50 per extra item added.
What is the optimal solution?

6–39 Show that, for a system of n components in series, the system reliability is the product of the component reliabilities.

6–40 Show that, for a system of n components in parallel, the system reliability is one minus the product of the component failure probabilities.

Review Problems

6–41 One faculty member is chosen from those summarized in Figure 6–15. All persons have the same chance of being selected. Determine the following conditional probabilities:
(a) Pr[CE | woman]
(b) Pr[EE | man]
(c) Pr [ME | man *and* FT]
(d) Pr[EE | woman *and* PT]
(e) Pr[FT | woman]
(f) Pr[FT *and* EE | man]
(g) Pr[man | CE *or* EE]
(h) Pr[PT *and* CE | man]

6–42 Consider the faculty data in Figure 6–15. A lottery is held to select school membership on the university social committee. Three persons will be selected. Find the following probability that
(a) exactly 2 men will be chosen
(b) at least 1 woman will be chosen
(c) all will be from the electrical engineering department
(d) at least 1 part-timer will be chosen

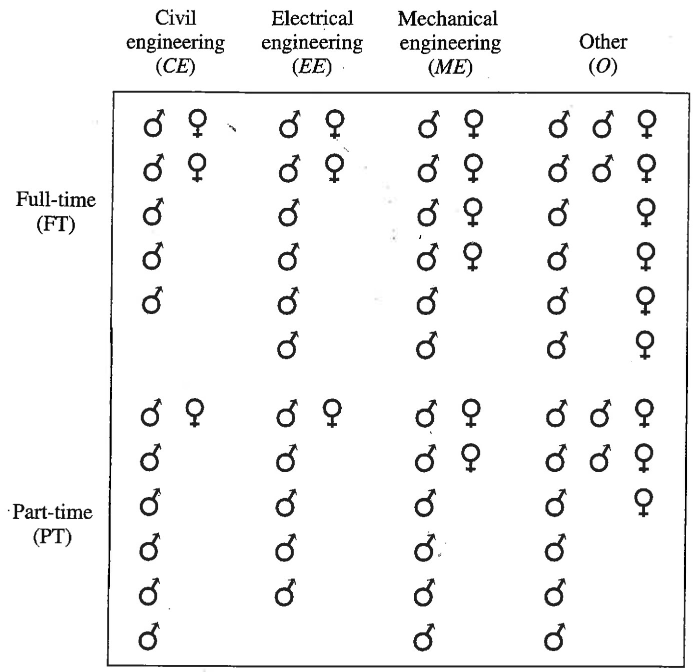


Figure 6–15 *Sample space for the characteristics of a randomly selected faculty member.*

6–43 The following facts are known regarding three events A, B, and C:

$$\Pr[A \text{ or } B] = \frac{3}{4} \qquad \Pr[A \text{ and } B] = \frac{1}{2} \qquad \Pr[B \mid C] = \frac{1}{2}$$

$$\Pr[B \text{ or } C] = \frac{3}{4} \qquad \Pr[A \text{ and } C] = \frac{1}{4} \qquad \Pr[A \mid B] = \frac{3}{4}$$

Find the following:

(a) $\Pr[B]$ (c) $\Pr[B \mid A]$ (e) $\Pr[B \text{ and } C]$
(b) $\Pr[A]$ (d) $\Pr[C]$ (f) $\Pr[C \mid B]$

6–44 The probability is .95 that a driver in a particular community will survive the year without an automobile accident. Assuming that accident experiences in successive years are independent events, find the probability that a particular driver

(a) goes 5 straight years with no accident
(b) has at least 1 accident in 5 years

6–45 Consider the elementary events for the sampling experiment in Figure 6–6. Apply the addition law to compute the following compound event probabilities:
(a) exactly 1 defective
(b) exactly 2 defectives
(c) at least 1 defective

6–46 Repeat Problem 6–45 using the experiment in Figure 6–5.

6–47 The probability of one or more power failures on an automobile assembly line is .10 during any given month. Assuming power events in successive months are independent, find the probability that there will be
(a) no power failures during a 3-month span of time
(b) exactly 1 month involving a power failure during the next 4 months
(c) at least 1 power failure during the next 5 months

6–48 Consider the elementary events for the sampling experiment in Figure 6–6. Determine the following probabilities:

(a) $\Pr[G_1 \text{ } and \text{ } G_2]$	(c) $\Pr[G_1 \mid G_2]$	(e) $\Pr[G_3]$
(b) $\Pr[G_2]$	(d) $\Pr[G_2 \text{ } and \text{ } G_3]$	(f) $\Pr[G_2 \mid G_3]$

6–49 Repeat Problem 6–48 using the experiment in Figure 6–5.

6–50 A sample of 4 parts is selected from a production line where 20% are overweight.
(a) Construct a probability tree diagram for this experiment in which each sample observation is treated as a separate stage.
(b) Find the probability that the number of overweights in the sample is

(1) zero	(3) exactly 2	(5) exactly 4
(2) exactly 1	(4) exactly 3	(6) at least 2

6–51 Suppose that a customer inspects a sample of 4 parts taken without replacement from a shipment of 20 parts from the above plant. Repeat (a) and (b) of Problem 6–50.