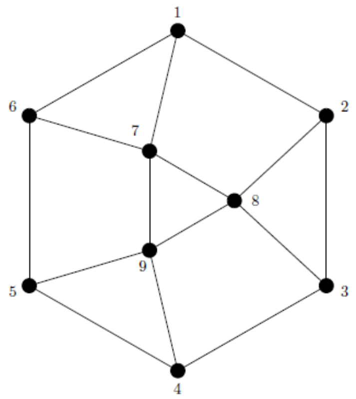
1 Markov chains and renewal processes

Consider a random walk X moving on the following graph. The walk starts at $X_0 = 1$ and at each step picks one of the neighbours of the current position uniformly at random and moves to it.



1. What is the equilibrium distribution, π?
2. What is the mean return time to state 1?
3. Write $O = \{1, 2, 3, 4, 5, 6\}$ for the states in the outer ring and $I = \{7, 8, 9\}$ for the states in the inner ring. Consider the sequence $H_0, H_1, H_2, \dots$ of times defined recursively by $H_0 = \inf\{n \geq 0 : X_n \in I\}$ and, for $m \geq 0$,
$$H_{m+1} = \inf\{n > H_m : X_n \in I\}.$$
Let $N(n) = \#\{m \geq 0 : H_m \leq n\}$ and consider the process $(N(n), n \geq 0)$. Explain why N is a delayed renewal process.
Hint: you may find it helpful to define $f(x) = I$ for $x \in I$ and $f(x) = O$ for $x \in O$ and then let $Y_n = f(X_n)$. What can you say about the process $(Y_n)_{n\geq 0}$?
4. Show that $H_1 - H_0$ has the same distribution as
$$1 + BG$$

where B and G are independent, $B \sim \text{Ber}(1/2)$ and $G \sim \text{Geom}(1/3)$ (i.e. $\mathbb{P}[G = k] = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$ for $k \geq 1$).

5. Suppose now that we instead start with $X_0 \sim \pi$. What is the probability mass function of H_0 in this case? (In other words, what is the special delay distribution that makes the renewal process stationary?) Deduce that H_0 has the same distribution as

$$B'G',$$

where B' and G' are a Bernoulli random variable and an independent Geometric random variable respectively, whose parameters you should determine.

2 Martingales and optional stopping

Let $S_n = \xi_1 + \cdots + \xi_n$ be a simple *asymmetric* random walk on $\mathbb{Z}$, started at zero, where $\mathbb{P}[\xi = 1] = p = 1 - \mathbb{P}[\xi = -1]$, for some $p \in (1/2, 1)$.

1. Define the function ϕ by $\phi(x) = \left(\frac{1-p}{p}\right)^x$. Show that $\phi(S_n)$ is a martingale.

2. Let $T_x := \inf\{n \geq 0 : S_n = x\}$. Prove that for any two levels $a < 0 < b$,

$$\mathbb{P}[T_a < T_b] = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)}.$$

(Hint: consider the stopping time $N = T_a \wedge T_b$, which you may assume is almost surely finite!)

3. Show that $S_n - (2p-1)n$ is a martingale. Use this to prove that $\mathbb{E}[T_b] = \frac{b}{2p-1}$. (You may assume that $\mathbb{E}[S_{T_b \wedge n}] \to \mathbb{E}[S_{T_b}]$ as $n \to \infty$.)

4. Show that the process X is a martingale, where

$$X_n = (S_n - (2p-1)n)^2 - 4p(1-p)n.$$

5. Use the previous two parts of this question to show that

$$\text{Var}[T_b] = \frac{4p(1-p)b}{(2p-1)^3}.$$

3 Foster-Lyapunov criteria

Consider the discrete-time Markov chain X on the non-negative integers, with transition probabilities for $x \geq 2$ given by

$$p_{x,x-2} = p_{x,x+1} = \frac{1}{2},$$

while if $X_n \in \{0, 1\}$ then

$$X_{n+1} = X_n + V_{n+1}$$

where $\{V_n\}_{n \geq 0}$ are a sequence of i.i.d. Poisson(1) random variables.

1. Show that $C = \{0, 1\}$ is a small set (of lag 1).

2. Use the Foster-Lyapunov criterion for geometric ergodicity to show that X is geometrically ergodic. (Hint: try using a scale function of the form $\Lambda(x) = \beta^x$, for an appropriate β.)