Remember from the previous section that there are other ways of stating conditional claims besides using "if . . . then. . . ." For example, this is another way to state the foregoing argument:

Example 2:

Madderly wins the decision provided the referee scored the fight in his favor.

The referee did score the fight in favor of Madderly.

Therefore, Madderly wins the decision.

Here are some other examples of arguments that fit the modus ponens form, accompanied by some remarks about why they do. Make sure you understand each one. (We'll continue to separate premises from the conclusion with the horizontal line; it works the same as "therefore" in introducing the conclusion.)

Example 3:

The generator works.

The generator works only if the polarity of the circuit has been reversed.

The polarity of the circuit has been reversed.

The second premise is the required conditional, with "only if" introducing the consequent (see page 298). It does not matter which of the two premises is stated first. The full conditional is stated second in this example.

Example 4:

Failure to melt at 2,600 degrees is sufficient for determining that this item is not made of steel.

The item failed to melt at 2,600 degrees.

The item is not made of steel.

The first premise is a conditional stated in terms of a sufficient condition (see page 301).

Modus Tollens

Modus tollens ("in the denial mode," approximately) is also a two-premise argument with one of the premises a conditional and the other premise the *negation* of that conditional's consequent. The conclusion is the negation of the antecedent. It looks like this:

If P then Q.

Not-Q.

Not-P.

It is crucially important to notice that the nonconditional premise is the negation of the consequent of the other premise, not its antecedent. It isn't valid the other way, as you'll see in the following examples. Every argument that fits the form above, however, is valid. You'll sometimes hear someone use modus tollens in this way: "Hey, if X had happened, then Y would have had to happen, but it (Y) didn't happen. So X must not have happened."

Here are some examples of modus tollens. Make sure you understand why they fit the modus tollens form.

Example 1:

If the new generator will work, then the polarity of the circuit has been reversed.

But the polarity of the circuit has not been reversed.

The new generator will not work.

Example 2:

If the song is in A-minor, there are no black keys in its scale.

However, there are black keys in its scale.

The song is not in A-minor.

Example 3:

If he got his forms in on time, he's automatically accepted. But he wasn't automatically accepted; consequently, he must not have gotten his forms in on time.

Example 4:

Bill was not with AT&T; but he'd have to have been with AT&T if he'd had an early iPhone. So he clearly did not have an early iPhone.

A last point about modus tollens: This argument form is the logical structure underlying the technique known as *"reductio ad absurdum"*—literally, to reduce to an absurdity. This technique, widely used in science and mathematics (where it's sometimes known as "indirect proof"), attempts to show that a given claim clearly leads to (implies) a second claim, and that the second claim cannot be true. Thus, by modus tollens, the first claim cannot be true either.

Chain Argument

A chain argument comprises two conditionals for premises and another for the conclusion. Here's the form:

If P then Q.

If Q then R.

If P then R.

The important thing is the arrangement of the simple sentences, P, Q, and R. It's crucial that the consequent of one premise be the same as the antecedent of the other premise. The remaining antecedent and consequent get hooked up in the conclusion. This way of laying it out may help:

If P . . . then Q.

If Q . . . then R.

If P . . . then R.

Here, Q is the consequent of the first premise and the antecedent of the second. This makes possible the conclusion with P as antecedent and R as consequent. Here are further examples of the chain argument:

Example 1:

If Casey goes to the meeting, then Simone will go.
If Simone goes, then Chris will go.
If Casey goes to the meeting, then Chris will go.

Example 2:

If the stock is lightweight, then it's aluminum; but if it's aluminum, then it will be hard to weld. So, if the stock is lightweight, it will be hard to weld.

Example 3:

If the picture is a daguerreotype, then it has to have been made after 1837; and the man in the picture can't be Hegel if it was made after 1837. So it isn't Hegel in the picture if it's a daguerreotype.

Example 4:

If the oil gusher in the gulf of Mexico continues until August, it will be ten times bigger than the Exxon Valdez spill. And if it's that much bigger than the Exxon Valdez, it'll be the biggest man-made environmental disaster ever. You draw the conclusion.

Three Mistakes: Invalid Argument Forms

Each of the three foregoing argument forms has an invalid imposter that resembles it fairly closely. We'll have a brief look at each and see why they fail the test of validity.

Affirming the Consequent

This fallacy masquerades as modus ponens, because one of its premises is a conditional and the other premise is a part of that conditional, while the conclusion is the other part of the conditional. But modus ponens, you'll recall, has the *antecedent* of the conditional as its other premise and its *consequent* as conclusion. The fallacious version has the consequent as the other premise and the antecedent of the conditional as conclusion. Thus the title, since the second premise "affirms" or states the consequent of the conditional rather than its antecedent. This is easy to see when we lay it out like this:

If P then Q.
Q.
Therefore, P.

You might take a moment to compare this argument form to the one for modus ponens on page 311. In this case, it is entirely possible for the premises to both be true and the conclusion false. If Q is true and P is false, then both premises are true* and the conclusion false—exactly the thing that cannot happen with

* Since a conditional with false antecedent and true consequent is true; remember the truth table for the conditional on page 298.

a valid argument. So remember, a conditional with its *antecedent* as the other premise can validly give the consequent as a conclusion. However, a conditional with its *consequent* as the other premise cannot validly produce the antecedent as conclusion.

Example 1:

If Shelley has read the *Republic*, then she's bound to know who Thrasymachus is. And, since she clearly *does* know who Thrasymachus is, we can conclude that she must have read the *Republic*.

Example 2:

This zinfandel would have a smooth finish if it came from very old vines. In fact, it does have a smooth finish, so it must have come from very old vines.

Denying the Antecedent

Here, the fallacy impersonates modus tollens. In modus tollens, a conditional and the negation of its consequent validly give us the negation of its antecedent. But the fallacious version has us trying to draw a conclusion from a conditional and the negation of its *antecedent*, and this does not produce a valid argument.

Here's the way it looks:

If P then Q.
Not-P.
Therefore, not-Q.

Now, if P is false and Q is true, the premises of the argument are true and the conclusion is false; thus the argument is invalid. A couple of examples:

Example 1:

If Jared studies really hard for the final, he will pass the course. [Later:] Well, Jared didn't study for the final, so it's a sure thing that he won't pass.

Example 2:

Joel will automatically be accepted provided he got his forms in on time. Unfortunately, he did not get his forms in on time; so he won't be automatically accepted.

Undistributed Middle (truth-functional version)

Our final invalid form mimics the chain argument. It differs from that argument, though, by having the same consequent for each of its conditional premises:

If P then Q.
If R then Q.
Therefore, if P then R.

If we allow P and Q to be true and R to be false, we'll see that both premises turn out to be true and the conclusion false. So this is clearly not a valid form of argument. Here's what it can look like:

Example 1:

If Robinson had had some great success in business, he'd be well-known. Furthermore, if he were extremely rich, he'd be well-known. So, of course, if Robinson had had some great success in business, he'd be extremely rich.

Example 2:

If you eat fish, you're a carnivore; and if you're an omnivore, you must be a carnivore. So, if you eat fish, that would make you an omnivore.

Exercise 9-4

Go through the eighteen examples in this section and symbolize each. Alternatively, use "if . . . then . . ." and "not- . . ." in place of the special symbols.

Exercise 9-5

For each of the last six examples (of the invalid forms), explain why each is invalid. For each, try to imagine and describe circumstances in which the premises are true and the conclusion is false.

Exercise 9-6

Determine which of the argument forms mentioned in this section is found in each of the following passages. Which contain valid arguments and which do not?

W & → T

1. There are Taliban in North Waziristan; and if there are Taliban there, you can be sure they're in South Waziristan, too. So we have to believe there are Taliban in South Waziristan.
2. If the Saints win the Super Bowl again, it will be poetic justice for New Orleans, the country's most bad-luck city in recent years. Unfortunately, the Saints have no chance to repeat the win, so there'll be no poetic justice this year for "N'awlins."
3. If you read Ayn Rand, you'll be a libertarian. And, of course, if you're an anarchist, you're already a libertarian. Hmm. It looks like if you read Ayn Rand, you'll be an anarchist!

S → T

4. If Sheila were ever to become a successful trader, she would have to develop a ruthless personality. But you know her: she could never be ruthless, even for a minute. So it's not going to be in her future to be a successful trader.
5. It's true, Ms. Zerkle will be accepted into law school only if she has excellent grades. But I'm telling you, you should see her transcript; she's made straight A's for the past two years. So don't worry about her getting into law school. She'll be accepted without a doubt.
6. If the Lambda Xi's continue to throw those open parties, they're going to get cited by the police. So if they continue the parties, they'll get decertified by the university because the university will certainly decertify them if they're cited by the police.

7. Jamal is a devout Muslim only if he follows the Sharia law, and I know for a fact that he follows it to the letter. So he is a devout Muslim.
8. If the carburetor is clogged, the engine will run lean, and running lean will lead to overheating. So overheating can result if the carburetor is clogged up.

More exercises related to this section can be found in Exercise 9-19, near the end of the chapter.

TRUTH-FUNCTIONAL ARGUMENTS

Categorical syllogisms (discussed in Chapter 8) have a total of 256 forms. A truth-functional argument, by contrast, can take any of an infinite number of forms. Nevertheless, we have methods for testing for validity that are flexible enough to encompass every truth-functional argument. In the remainder of this chapter, we'll look at three of them: the truth-table method, the short truth-table method, and the method of deduction.

Before doing anything else, though, let's quickly review the concept of validity. An argument is *valid*, you'll recall, if and only if the truth of the premises guarantees the truth of the conclusion—that is, if the premises were true, the conclusion could not then be false. (Where validity is concerned, remember, it doesn't matter whether the premises are *actually* true.)

The Truth-Table Method

The *truth-table test for validity* requires familiarity with the truth tables for the four truth-functional symbols, so go back and check yourself on those if you think you may not understand them clearly. Here's how the method works: We present all of the possible circumstances for an argument by building a truth table for it; then we simply look to see if there are any circumstances in which the premises are all true and the conclusion false. If there are such circumstances—one row of the truth table is all that's required—then the argument is invalid.

Let's look at a simple example. Let P and Q represent any two claims. Now, look at the following symbolized argument:

P → Q
~P
Therefore, ~Q

We can construct a truth table for this argument by including a column for each premise and one for the conclusion:

1 P	2 Q	3 ~P	4 P → Q	5 ~Q
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

The first two columns are reference columns; they list truth values for the letters that appear in the argument. The reference columns should be constructed in accordance with the method described on page 301. The third and fourth columns appear under the two premises of the argument, and the fifth column is for the conclusion. The truth values in these columns are determined by those in the appropriate rows of the reference columns. Note that in the third row of the table, both premises are true and the conclusion is false. This tells us that it is possible for the premises of this argument to be true while the conclusion is false; thus, the argument is invalid. Because it doesn't matter what claims P and Q might stand for, the same is true for *every* argument of this pattern. Here's an example of such an argument:

> If the Saints beat the Forty-Niners, then the Giants will make the playoffs. But the Saints won't beat the Forty-Niners. So the Giants won't make the playoffs.

Using S for "The Saints beat (or will beat) the Forty-Niners" and G for "The Giants make (or will make) the playoffs," we can symbolize the argument like this:

$S \rightarrow G$ → argument form
$\sim S$
$\overline{\sim G}$

invalid

The first premise is a conditional, and the other premise is the negation of the antecedent of that conditional. The conclusion is the negation of the conditional's consequent. It has exactly the same structure as the argument for which we just did the truth table; accordingly, it, too, is invalid.

Let's do another simple one:

> We're going to have large masses of arctic air (A) flowing into the Midwest unless the jet stream (J) moves south. Unfortunately, there's no chance of the jet stream's moving south. So you can bet there'll be arctic air flowing into the Midwest.

Symbolization gives us

$A \vee J$
$\sim J$
$\overline{A}$

Here's a truth table for the argument:

valid

1 A	2 J	3 A ∨ J	4 ~J
T	T	T	F
T	F	T	T
F	T	T	F
F	F	F	T

Note that the first premise is represented in column 3 of the table, the second premise in column 4, and the conclusion in one of the reference columns, column 1. Now, let's recall what we're up to. We want to know whether this argument is valid—that is to say, is it possible for the premises to be true and the conclusion false? If there is such a possibility, it will turn up in the truth table because, remember, the truth table represents every possible situation with respect to the claims A and J. We find that the premises are both true in only one row, the second, and when we check the conclusion, A, we find it is true in that row. Thus, there is *no* row in which the premises are true and the conclusion false. So, the argument is valid.

Here's an example of a rather more complicated argument:

> If Scarlet is guilty of the crime, then Ms. White must have left the back door unlocked and the colonel must have retired before ten o'clock. However, either Ms. White did not leave the back door unlocked, or the colonel did not retire before ten. Therefore, Scarlet is not guilty of the crime.

Let's assign some letters to the simple claims so that we can show this argument's pattern.

S = Scarlet is guilty of the crime.
W = Ms. White left the back door unlocked.
C = The colonel retired before ten o'clock.

Now we symbolize the argument to display this pattern:

$S \rightarrow (W \,\&\, C)$
$\sim W \vee \sim C$
$\overline{\sim S}$

Let's think our way through this argument. As you read, refer back to the symbolized version above. Notice that the first premise is a conditional, with "Scarlet is guilty of the crime" as antecedent and a conjunction as consequent. In order for that conjunction to be true, both "Ms. White left the back door unlocked" and "The colonel retired before ten o'clock" have to be true, as you'll recall from the truth table for conjunctions. Now look at the second premise. It is a disjunction that tells us *either* Ms. White did not leave the back door unlocked *or* the colonel did not retire before ten. But if either or both of those disjuncts are true, at least one of the claims in our earlier conjunction is false. So it cannot be that *both* parts of the conjunction are true. This means the conjunction symbolized by W & C must be false. And so the consequent of the first premise is false. How can the entire premise be true, in that case? The only way is for the antecedent to be false as well. And that means that the conclusion, "Scarlet is not guilty of the crime," must be true.

All of this reasoning (and considerably more that we don't require) is implicit in the following truth table for the argument:

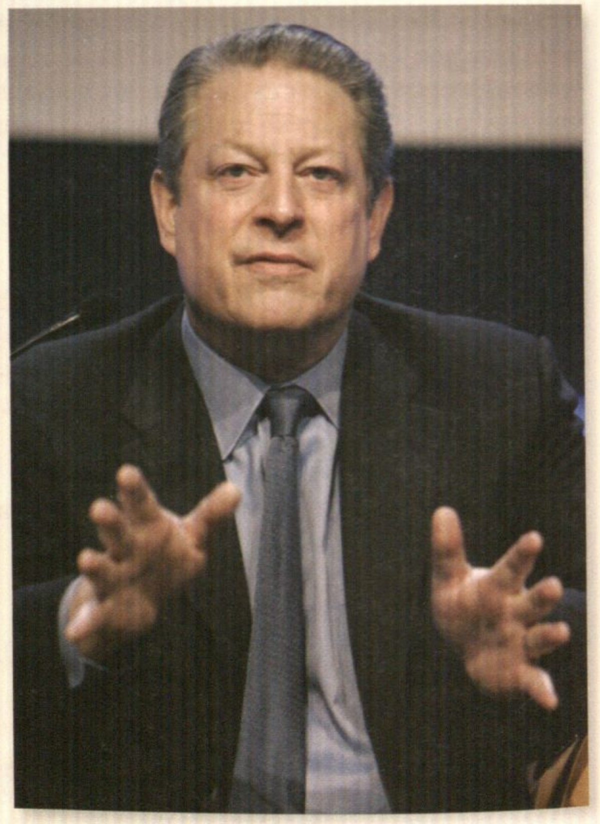
Real Life

An Al Gore Chain Argument

If governments (which have not indicated they are willing to act) do not act soon, carbon emissions will cause a serious increase in global temperatures. And if that happens, there will be a series of planetary catastrophes. Finally, if these catastrophes take place, the world will become uninhabitable. So, unfortunately, our ability to live on this planet depends on the timely actions of governments that so far have shown little inclination to act.

—Our exaggeration of an Al Gore thesis

Notice that this passage is simply two chain arguments linked together.



1 S	2 W	3 C	4 ~W	5 ~C	6 W & C	7 S → (W & C)	8 ~W ∨ ~C	9 ~S
T	T	T	F	F	T	T	F	F
T	T	F	F	T	F	F	T	F
T	F	T	T	F	F	F	T	F
T	F	F	T	T	F	F	T	F
F	T	T	F	F	T	T	F	T
F	T	F	F	T	F	T	T	T
F	F	T	T	F	F	T	T	T
F	F	F	T	T	F	T	T	T

The first three columns are our reference columns, columns 7 and 8 are for the premises of the argument, and column 9 is for the argument's conclusion. The remainder—4, 5, and 6—are for parts of some of the other symbolized claims; they could be left out if we desired, but they make filling in columns 7 and 8 a bit easier.

Once the table is filled in, evaluating the argument is easy. Just look to see whether there is any row in which the premises are true and the conclusion is false. One such row is enough to demonstrate the invalidity of the argument.

In the present case, we find that both premises are true only in the last three rows of the table. And in those rows, the conclusion is also true. So there is no set of circumstances—no row of the table—in which both premises are true and the conclusion is false. Therefore, the argument is valid.

The Short Truth-Table Method

Although filling out a complete truth table always produces the correct answer regarding a truth-functional argument's validity, it can be quite a tedious chore—in fact, life is much too short to spend much of it filling in truth tables. Fortunately, there are shorter and more manageable ways of finding such an answer. The easiest systematic way to determine the validity or invalidity of truth-functional arguments is the *short truth-table method.* Here's the idea behind it: *If an argument is invalid, there has to be at least one row in the argument's truth table where the premises are true and the conclusion is false.* With the short truth-table method, we simply focus on finding such a row. Consider this symbolized argument:

$$\begin{array}{l} P \to Q \\ \sim Q \to R \\ \hline \sim P \to R \end{array}$$

We begin by looking at the conclusion. Because it's a conditional, it can be made false only one way, by making its antecedent true and its consequent false. So, we do that by making P false and R false.

Can we now make both premises true? Yes, as it turns out, by making Q true. This case,

P	Q	R
F	T	F

makes both premises true and the conclusion false and thus proves the argument invalid. What we've done is produce the relevant row of the truth table without bothering to produce all the rest. Had the argument been valid, we would not have been able to produce such a row.

Here's how the method works with a valid argument. Consider this example:

$$\begin{array}{l} (P \vee Q) \to R \\ S \to Q \\ \hline S \to R \end{array}$$

The only way to make the conclusion false is to make S true and R false. So, we do that:

P	Q	R	S
		F	T

Now, with S true, the only way we can make the second premise true is by making Q true. So, we do that next:

P	Q	R	S
	T	F	T

But now, there is no way at all to make the first premise true, because P ∨ Q is going to be true (because Q is true), and R is already false. Because there is no other way to make the conclusion false and the second premise true, and because this way fails to make the first premise true, we can conclude that the argument is *valid*.

In many cases, there will be more than one way to make the conclusion false. Here's a symbolized example:

$$\left.\begin{array}{l} P \mathbin{\&} (Q \vee R) \\ R \rightarrow S \\ P \rightarrow T \end{array}\right\} \text{trying to make these true}$$
$$\left.\begin{array}{l} \hline S \mathbin{\&} T \end{array}\right\} \text{trying to make this false}$$

Because the conclusion is a conjunction, it is false if either or both of its conjuncts are false, which means we could begin by making S true and T false, S false and T true, or both S and T false. This is trouble we'd like to avoid if possible, so let's see if there's someplace else we can begin making our assignment. (Remember: The idea is to try to assign true and false to the letters so as to make the premises true and the conclusion false. If we can do it, the argument is invalid.)

In this example, to make the first premise true, we *must* assign true to the letter P. Why? Because the premise is a conjunction, and both of its parts must be true for the whole thing to be true. That's what we're looking for: places where we are *forced* to make an assignment of true or false to one or more letters. Then we make those assignments and see where they lead us. In this case, once we've made P true, we see that, to make the third premise true, we are forced to make T true (because a true antecedent and a false consequent would make the premise false, and we're trying to make our premises true).

After making T true, we see that, to make the conclusion false, S must be false. So we make that assignment. At this point we're nearly done, needing only assignments for Q and R.

P	Q	R	S	T
T			F	T

Are there any other assignments that we're forced to make? Yes: We must make R false to make the second premise true. Once we've done that, we see that Q must be true to preserve the truth of the first premise. And that completes the assignment:

P	Q	R	S	T
T	T	F	F	T

This is one row in the truth table for this argument—the only row, as it turned out—in which all the premises are true and the conclusion is false; thus, it is the row that proves the argument invalid.

In the preceding example, there was a premise that forced us to begin with a particular assignment to a letter. Sometimes, neither the conclusion nor any of the premises forces an assignment on us. In that case, we must use trial and error: Begin with one assignment that makes the conclusion false (or some premise true) and see if it will work. If not, try another assignment. If all fail, then the argument is valid.

Often, several rows of a truth table will make the premises true and the conclusion false; any one of them is all it takes to prove invalidity. Don't get the mistaken idea that, just because the premises are all true in one row and so is the conclusion, the conclusion follows from the premises—that is, that the argument must be valid. To be valid, the conclusion must be true in *every* row in which all the premises are true.

To review: Try to assign Ts and Fs to the letters in the symbolization so that all premises come out true and the conclusion comes out false. There may be more than one way to do it; any of them will do to prove the argument invalid. If it is impossible to make the premises and conclusion come out this way, the argument is valid.

Exercise 9-7

Construct full truth tables or use the short truth-table method to determine which of the following arguments are valid.

▲ 1. P ∨ ~Q
~Q
―――
~P

2. P → Q
~Q
―――
~P

3. ~(P ∨ Q)
R → P
―――
~R

▲ 4. P → (Q → R)
~(P → Q)
―――
R

5. P ∨ (Q → R)
Q & ~R
―――
~P

6. (P → Q) ∨ (R → Q)
P & (~P → ~R)
―――
Q

▲ 7. $(P \& R) \to Q$
$\sim Q$
———
$\sim P$

8. $P \& (\sim Q \to \sim P)$
$R \to \sim Q$
———
$\sim R$

9. $L \vee \sim J$
$R \to J$
———
$L \to \sim R$

10. $\sim F \vee (G \& H)$
$P \to F$
———
$\sim H \to \sim P$

Exercise 9-8

Use either the long or short truth-table method to determine which of the following arguments are valid.

▲ 1. $K \to (L \& G)$
$M \to (J \& K)$
$B \& M$
———
$B \& G$

▲ 2. $L \vee (W \to S)$
$P \vee \sim S$
$\sim L \to W$
———
P

▲ 3. $M \& P$
$R \to \sim P$
$F \vee R$
$G \to M$
———
$G \& F$

▲ 4. $(D \& G) \to H$
$M \& (H \to P)$
$M \to G$
———
$D \& P$

▲ 5. $R \to S$
$(S \& B) \to T$
$T \to E$
———
$(R \vee B) \to E$

DEDUCTIONS

The next method we'll look at is less useful for proving an argument *invalid* than the truth-table methods, but it has some advantages in proving that an argument is valid. The method is that of **deduction.**

When we use this method, we actually deduce (or "derive") the conclusion from the premises by means of a series of basic, truth-functionally valid argument patterns. This is a lot like "thinking through" the argument, taking one step at a time to see how, once we've assumed the truth of the premises, we eventually arrive at the conclusion. (We do this for an example on page 319.) We'll consider some extended examples showing how the method works as we explain the first few basic argument patterns. We'll refer to these patterns as truth-functional rules because they govern what steps we're allowed to take in getting from the premises to the conclusion. (Your instructor may ask that you learn some or all of the basic valid argument patterns. It's a good idea to be able to identify these patterns whether you go on to construct deductions from them or not.)

Group I Rules: Elementary Valid Argument Patterns

This first group of rules should be learned before you go on to the Group II rules. Study them until you can work Exercise 9-9 with confidence.

Rule 1: Modus ponens (MP), also known as *affirming the antecedent*

Any argument of the pattern

$P \to Q$
P
———
Q

is valid. If you have a conditional among the premises, and if the antecedent of that conditional occurs as another premise, then by **modus ponens** the consequent of the conditional follows from those two premises. The claims involved do not have to be simple letters standing alone—it would have made no difference if, in place of P, we had had something more complicated, such as $(P \vee R)$, as long as that compound claim appeared everywhere that P appears in the pattern above. For example:

1.	$(P \vee R) \to Q$	Premise
2.	$P \vee R$	Premise
3.	Q	From the premises, by modus ponens

The idea, once again, is that if you have *any conditional whatsoever* on a line of your deduction, and if you have the antecedent of that conditional on some other line, you can write down the consequent of the conditional on your new line.

If the consequent of the conditional is the conclusion of the argument, then the deduction is finished—the conclusion has been established. If it is not the conclusion of the argument you're working on, the consequent of the conditional can be listed just as if it were another premise to use in deducing the conclusion you're after. An example:

1. $P \to R$
2. $R \to S$
3. P Therefore, S

We've numbered the three premises of the argument and set its conclusion off to the side. (Hereafter we'll use a slash and three dots [$/\therefore$] in place of "therefore" to indicate the conclusion.) Now, notice that line 1 is a conditional, and line 3 is its antecedent. Modus ponens allows us to write down the consequent of line 1 as a new line in our deduction:

4. R 1, 3, MP

At the right, we've noted the abbreviation for the rule we used and the lines the rule required. These notes are called the *annotation* for the deduction. We can now make use of this new line in the deduction to get the conclusion we were originally after, namely, S.

5. S 2, 4, MP

Again, we used modus ponens, this time on lines 2 and 4. The same explanation as that for deriving line 4 from lines 1 and 3 applies here.

Notice that the modus ponens rule and all other Group I rules can be used only on whole lines. This means that you can't find the items you need for MP as *parts* of a line, as in the following:

$$\begin{array}{l} (P \to Q) \vee R \\ P \\ \hline Q \vee R \end{array} \qquad \text{(erroneous!)}$$

This is *not* a legitimate use of MP. We do have a conditional as *part* of the first line, and the second line is indeed the antecedent of that conditional. But the rule cannot be applied to parts of lines. The conditional required by rule MP must take up the entire line, as in the following:

$$\begin{array}{l} P \to (Q \vee R) \\ P \\ \hline Q \vee R \end{array}$$

Rule 2: Modus tollens (MT), also known as *denying the consequent*

The **modus tollens** pattern is this:

$$\begin{array}{l} P \to Q \\ \sim Q \\ \hline \sim P \end{array}$$

If you have a conditional claim as one premise and if one of your other premises is the negation of the consequent of that conditional, you can write down the negation of the conditional's antecedent as a new line in your deduction. Here's a deduction that uses both of the first two rules:

1. $(P \& Q) \to R$
2. S
3. $S \to \sim R$ $/\therefore \sim(P \& Q)$
4. $\sim R$ 2, 3, MP
5. $\sim(P \& Q)$ 1, 4, MT

In this deduction, we derived line 4 from lines 2 and 3 by modus ponens, and then 4 and 1 gave us line 5, which is what we were after, by modus tollens. The fact that the antecedent of line 1 is itself a compound claim, (P & Q), is not important; our line 5 is the antecedent of the conditional with a negation sign in front of it, and that's all that counts.

Real Life

If the Dollar Falls . . .

The valid argument patterns are in fact fairly common. Here's one from an article in *Time* as to why a weakening dollar is a threat to the stock market:

> Why should we care? . . . If the dollar continues to drop, investors may be tempted to move their cash to currencies on the upswing. That would drive the U.S. market lower. . . . Because foreigners hold almost 40% of U.S. Treasury securities, any pullout would risk a spike in interest rates that would ultimately slaughter the . . . market.

The chain argument here is reasonably obvious. In effect: If the dollar falls, then investors move their cash to currencies on the upswing. If investors move their cash to currencies on the upswing, then the U.S. market goes lower. If the U.S. market goes lower, then interest rates on U.S. Treasury securities rise. If interest rates on U.S. Treasury securities rise, then the . . . market dies. [Therefore, if the dollar falls, then the . . . market dies.]

Rule 3: Chain argument (CA)

$$\begin{array}{l} P \to Q \\ Q \to R \\ \hline P \to R \end{array}$$

The **chain argument** rule allows you to derive a conditional from two you already have, provided the antecedent of one of your conditionals is the same as the consequent of the other.

Rule 4: Disjunctive argument (DA)

$$\begin{array}{l} P \vee Q \\ \sim P \\ \hline Q \end{array} \qquad \begin{array}{l} P \vee Q \\ \sim Q \\ \hline P \end{array}$$

From a disjunction and the negation of one disjunct, the other disjunct may be derived.

Rule 5: Simplification (SIM)

This one is obvious, but we need it for obvious reasons:

$$\begin{array}{l} P \& Q \\ \hline P \end{array} \qquad \begin{array}{l} P \& Q \\ \hline Q \end{array}$$

If the conjunction is true, then of course the conjuncts must all be true. You can pull out one conjunct from any conjunction and make it the new line in your deduction.

Rule 6: Conjunction (CONJ)

$$\begin{array}{l} P \\ Q \\ \hline P \& Q \end{array}$$

This rule allows you to put any two lines of a deduction together in the form of a conjunction.

Rule 7: Addition (ADD)

$$\frac{P}{P \vee Q} \qquad \frac{Q}{P \vee Q}$$

Clearly, no matter what claims P and Q might be, if P is true then *either* P or Q must be true. The truth of one disjunct is all it takes to make the whole disjunction true.

Rule 8: Constructive dilemma (CD)

$$\begin{array}{l} P \rightarrow Q \\ R \rightarrow S \\ \underline{P \vee R} \\ Q \vee S \end{array}$$

The disjunction of the antecedents of any two conditionals allows the derivation of the disjunction of their consequents.

Rule 9: Destructive dilemma (DD)

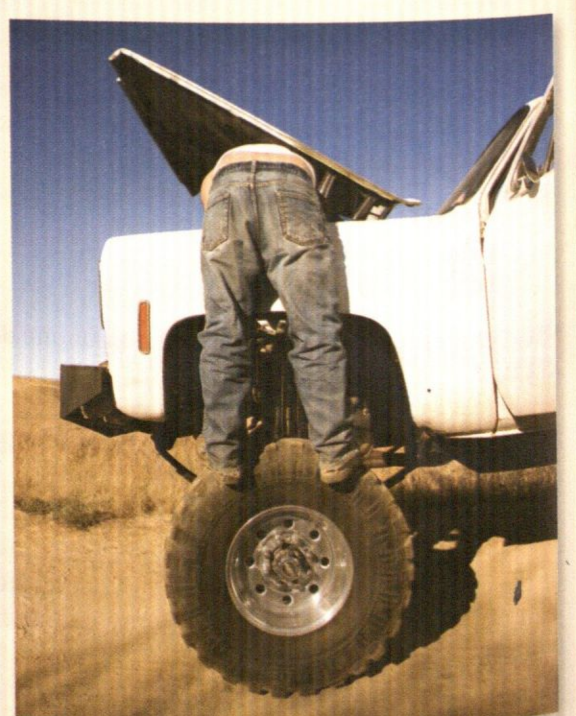
$$\begin{array}{l} P \rightarrow Q \\ R \rightarrow S \\ \underline{\sim Q \vee \sim S} \\ \sim P \vee \sim R \end{array}$$

The disjunction of the negations of the consequents of two conditionals allows the derivation of the disjunction of the negations of their antecedents. (Refer to the pattern above as you read this, and it will make a lot more sense.)

Real Life

Logician at Work

No, really. Problem solving in matters like auto mechanics involves a great deal of deductive reasoning. For example, "The problem had to be either a clogged fuel filter or a defective fuel pump. But we've replaced the fuel filter, and it wasn't that, so it has to be a bad fuel pump." This is an example of one of our Group I rules.



Exercise 9-9

For each of the following groups of symbolized claims, identify which Group I rule was used to derive the last line.

▲ 1. $P \rightarrow (Q \& R)$
$(Q \& R) \rightarrow (S \vee T)$
$P \rightarrow (S \vee T)$

▲ 2. $(P \& S) \vee (T \rightarrow R)$
$\sim(P \& S)$
$T \rightarrow R$

▲ 3. $P \vee (Q \& R)$
$(Q \& R) \rightarrow S$
$P \rightarrow T$
$S \vee T$

▲ 4. $(P \vee R) \rightarrow Q$
$\sim Q$
$\sim(P \vee R)$

▲ 5. $(Q \rightarrow T) \rightarrow S$
$\sim S \vee \sim P$
$R \rightarrow P$
$\sim(Q \rightarrow T) \vee \sim R$

Exercise 9-10

Construct deductions for each of the following, using the Group I rules. Each can be done in just a step or two (except number 10, which takes more).

▲ 1. 1. $R \rightarrow P$
2. $Q \rightarrow R$ $\quad /\therefore Q \rightarrow P$

2. 1. $P \rightarrow S$
2. $P \vee Q$
3. $Q \rightarrow R$ $\quad /\therefore S \vee R$

3. 1. $R \& S$
2. $S \rightarrow P$ $\quad /\therefore P$

▲ 4. 1. $P \rightarrow Q$
2. $\sim P \rightarrow S$
3. $\sim Q$ $\quad /\therefore S$

5. 1. $(P \vee Q) \rightarrow R$
2. Q $\quad /\therefore R$

6. 1. $\sim P$
2. $\sim(R \& S) \vee Q$
3. $\sim P \rightarrow \sim Q$ $\quad /\therefore \sim(R \& S)$

7. 1. $\sim S$
   2. $(P \& Q) \rightarrow R$
   3. $R \rightarrow S$ $\quad /\therefore \sim(P \& Q)$

8. 1. $P \rightarrow \sim(Q \& T)$
   2. $S \rightarrow (Q \& T)$
   3. P $\quad /\therefore \sim S$

9. 1. $(P \vee T) \rightarrow S$
   2. $R \rightarrow P$
   3. $R \vee Q$
   4. $Q \rightarrow T$ $\quad /\therefore S$

10. 1. $(T \vee M) \rightarrow \sim Q$
    2. $(P \rightarrow Q) \& (R \rightarrow S)$
    3. T $\quad /\therefore \sim P$

Group II Rules: Truth-Functional Equivalences

These rules are different from our Group I rules in some important ways. First, they are expressed as truth-functional equivalences. This means that they each take the form of two types of symbolizations that have exactly the same truth table. We'll use a double-headed arrow, $\leftrightarrow$, to indicate that we can move from either side to the other. (Remember that Group I rules allow us to go only one direction, from premises to conclusion.) A second major difference is that these rules can be used on *parts* of lines. So, if we have a conjunction in a deduction, and we have a Group II rule that says one of the conjuncts is equivalent to something else, we can substitute that something else for the equivalent conjunct. You'll see how this works after an example or two.

Here is the overall principle that governs how Group II rules work: *A claim or part of a claim may be replaced by a claim to which it is equivalent by one of the following Group II rules.* Once again, how this works should become clear in a moment. As in the case of the first group, the Ps and Qs and so forth in the statement of the rules can stand for any symbolized claim whatever, as long as each letter stands for the same claim throughout.

Rule 10: Double negation (DN)

$$P \leftrightarrow \sim\sim P$$

This rule allows you to add or remove two negation signs in front of any claim, whether simple or compound. For example, this rule allows the derivation of either of the following from the other,

$$P \rightarrow (Q \vee R) \qquad P \rightarrow \sim\sim(Q \vee R)$$

because the rule guarantees that $(Q \vee R)$ and its double negation, $\sim\sim(Q \vee R)$, are equivalent. This in turn guarantees that $P \rightarrow (Q \vee R)$ and $P \rightarrow \sim\sim(Q \vee R)$ are equivalent, and hence that each implies the other.

Here's an example of DN at work:

1. $P \vee \sim(Q \rightarrow R)$
2. $(Q \rightarrow R)$ $\quad /\therefore P$

3. $\sim\sim(Q \rightarrow R)$ $\quad$ 2, DN
4. P $\quad$ 1, 3, DA

Rule 11: Commutation (COM)

$$(P \& Q) \leftrightarrow (Q \& P)$$
$$(P \vee Q) \leftrightarrow (Q \vee P)$$

This rule allows any conjunction or disjunction to be "turned around" so that the conjuncts or disjuncts occur in reverse order. Here's an example:

$$P \rightarrow (Q \vee R) \qquad P \rightarrow (R \vee Q)$$

Either of these symbolized claims can be deduced from the other. Notice that commutation is used on *part* of the claim—just the consequent.

Rule 12: Implication (IMPL)

This rule allows us to change a conditional into a disjunction and vice versa.

$$(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$$

Notice that the antecedent always becomes the negated disjunct or vice versa, depending on which way you're going. Another example:

$$(P \vee Q) \rightarrow R \ \sim(P \vee Q)\ R$$

Rule 13: Contraposition (CONTR)

This rule may remind you of the categorical operation of contraposition (see Chapter 8)—this rule is its truth-functional version.

$$(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)$$

This rule allows us to exchange the places of a conditional's antecedent and consequent but only by putting on or taking off a negation sign in front of each. Here's another example:

$$(P \& Q) \rightarrow (P \vee Q) \ \sim(P \vee Q) \rightarrow \sim(P \& Q)$$

Sometimes you want to perform contraposition on a symbolization that doesn't fit either side of the equivalence because it has a negation sign in front of either the antecedent or the consequent but not both. You can do what you want in such cases, but it takes two steps, one applying double negation and one applying contraposition. Here's an example:

$(P \vee Q) \rightarrow \sim R$
$\sim\sim(P \vee Q) \rightarrow \sim R$ $\quad$ Double negation
$R \rightarrow \sim(P \vee Q)$ $\quad$ Contraposition

Your instructor may allow you to combine these steps (and refer to both DN and CONTR in your annotation).

Rule 14: DeMorgan's Laws (DEM)

$$\sim(P \& Q) \leftrightarrow (\sim P \vee \sim Q)$$
$$\sim(P \vee Q) \leftrightarrow (\sim P \& \sim Q)$$

Notice that, when the negation sign is "moved inside" the parentheses, the "&" changes into a "$\vee$," or vice versa. It's important not to confuse the use of the negation sign in DeMorgan's Laws with that of the minus sign in algebra. Notice that when you take $\sim(P \vee Q)$ and "move the negation sign in," you do *not* get $(\sim P \vee \sim Q)$. The wedge must be changed to an ampersand or vice versa whenever DEM is used. You can think of $\sim(P \vee Q)$ and $(\sim P \& \sim Q)$ as saying "neither P nor Q," and you can think of ~(P & Q) and (~P ∨ ~Q) as saying "not both P and Q."

Rule 15: Exportation (EXP)

$[P \to (Q \to R)] \leftrightarrow [(P \& Q) \to R]$

Square brackets are used exactly as parentheses are. In English, the exportation rule says that "If P, then if Q, then R" is equivalent to "If both P and Q, then R." (The commas are optional in both claims.) If you look back to Exercise 9-2, items 3 and 5 (page 310), you'll notice that, according to the exportation rule, each of these can replace the other.

Rule 16: Association (ASSOC)

$[P \& (Q \& R)] \leftrightarrow [(P \& Q) \& R]$
$[P \vee (Q \vee R)] \leftrightarrow [(P \vee Q) \vee R]$

Association simply tells us that, when we have three items joined together with wedges or with ampersands, it doesn't matter which ones we group together. If we have a long disjunction with more than two disjuncts, it still requires only one of them to be true for the entire disjunction to be true; if it's a conjunction, then all the conjuncts have to be true, no matter how many of them there are, in order for the entire conjunction to be true. Your instructor may allow you to drop parentheses in such symbolizations, but if you're developing these rules as a formal system, he or she may not.

Rule 17: Distribution (DIST)

This rule allows us to "spread a conjunct across a disjunction" or to "spread a disjunct across a conjunction." In the first example below, look at the left-hand side of the equivalence. The P, which is conjoined with a disjunction, is picked up and dropped (distributed) across the disjunction by being conjoined with each part. (This is easier to understand if you see it done on a chalkboard than by trying to figure it out from the page in front of you.) The two versions of the rule, like those of DEM, allow us to do exactly with the wedge what we're allowed to do with the ampersand.

$[P \& (Q \vee R)] \leftrightarrow [(P \& Q) \vee (P \& R)]$
$[P \vee (Q \& R)] \leftrightarrow [(P \vee Q) \& (P \vee R)]$

Rule 18: Tautology (TAUT)

$(P \vee P) \leftrightarrow P$
$(P \& P) \leftrightarrow P$

This rule allows a few obvious steps; they are sometimes necessary to "clean up" a deduction.

The twelve-step and seven-step examples that follow show some deductions that use rules from both Group I and Group II. Look at them carefully, covering up the lines with a piece of paper and uncovering them one at a time as you progress. This gives you a chance to figure out what you might do before you see the answer. In any case, make sure you understand how each line was achieved before going on. If necessary, look up the rule used to make sure you understand it.

The first example is long but fairly simple. Length is not always proportional to difficulty.

1. $P \to (Q \to R)$
2. $(T \to P) \& (S \to Q)$
3. $T \& S$ /∴ R

4.	$T \to P$	2, SIM
5.	$S \to Q$	2, SIM
6.	T	3, SIM
7.	S	3, SIM
8.	P	4, 6, MP
9.	Q	5, 7, MP
10.	P & Q	8, 9, CONJ
11.	$(P \& Q) \to R$	1, EXP
12.	R	10, 11, MP

Group I

1. Modus ponens (MP) P → Q P ——— Q	2. Modus tollens (MT) P → Q ~Q ——— ~P	3. Chain argument (CA) P → Q Q → R ——— P → R
4. Disjunctive argument (DA) P ∨ Q P ∨ Q ~P ~Q ——— Q P	5. Simplification (SIM) P & Q P & Q ——— P Q	6. Conjunction (CONJ) P Q ——— P & Q
7. Addition (ADD) P Q ——— P ∨ Q P ∨ Q	8. Constructive dilemma (CD) P → Q R → S P ∨ R ——— Q ∨ S	9. Destructive dilemma (DD) P → Q R → S ~Q ∨ ~S ——— ~P ∨ ~R

Group II

10. Double negation (DN) P ↔ ~~P	11. Commutation (COM) (P & Q) ↔ (Q & P) (P ∨ Q) ↔ (Q ∨ P)	12. Implication (IMPL) (P → Q) ↔ (~P ∨ Q)
13. Contraposition (CONTR) (P → Q) ↔ (~Q → ~P)	14. DeMorgan's Laws (DEM) ~(P & Q) ↔ (~P ∨ ~Q) ~(P ∨ Q) ↔ (~P & ~Q)	15. Exportation (EXPORT) [P → (Q → R)] ↔ [(P & Q) → R]
16. Association (ASSOC) [P & (Q & R)] ↔ [(P & Q) & R] [P ∨ (Q ∨ R)] ↔ [(P ∨ Q) ∨ R]	17. Distribution (DIST) [P & (Q ∨ R)] ↔ [(P & Q) ∨ (P & R)] [P ∨ (Q & R)] ↔ [(P ∨ Q) & (P ∨ R)]	18. Tautology (TAUT) (P ∨ P) ↔ P (P & P) ↔ P

FIGURE 2 Truth-Functional Rules for Deductions

It's often difficult to tell how to proceed when you first look at a deduction problem. One strategy is to work backward. Look at what you want to get, look at what you have, and see what you would need in order to get what you want. Then determine where you would get *that*, and so on. We'll explain in terms of the following problem.

1. P → (Q & R)		
2. S → ~Q		
3. S		/∴~P
4. ~Q	2, 3,	MP
5. ~Q ∨ ~R	4,	ADD
6. ~(Q & R)	5,	DEM
7. ~P	1, 6,	MT

We began by wanting ~P as our conclusion. If we're familiar with modus tollens, it's clear from line 1 that we can get ~P if we can get the negation of line 1's consequent, which would be ~(Q & R). That in turn is the same as ~Q ∨ ~R, which we can get if we can get either ~Q or ~R. So now we're looking for some place in the first three premises where we can get ~Q. That's easy: from lines 2 and 3, by modus ponens. A little practice and you'll be surprised how easy these strategies are to use, at least *most* of the time!

Exercise 9-11

The annotations that explain how each line was derived have been left off the following deductions. For each line, supply the rule used and the numbers of any earlier lines the rule requires.

▲ 1.
1. P → Q (Premise)
2. R → S (Premise)
3. Q → ~S (Premise) / ∴ P → ~R
4. P → ~S
5. ~S → ~R
6. P → ~R

2.
1. ~P (Premise)
2. (Q → R) & (R → Q) (Premise)
3. R ∨ P (Premise) / ∴ Q
4. R
5. R → Q
6. Q

3.
1. P → Q (Premise)
2. R → (~S ∨ T) (Premise)
3. ~P → R (Premise) / ∴ (~Q & S) → T
4. ~Q → ~P
5. ~Q → R
6. ~Q → (~S ∨ T)
7. ~Q → (S → T)
8. (~Q & S) → T

▲ 4.
1. (P & Q) → T (Premise)
2. P (Premise)
3. ~Q → ~P (Premise) / ∴ T
4. P → Q
5. Q
6. P & Q
7. T

5.
1. ~(S ∨ R) (Premise)
2. P → S (Premise)
3. T → (P ∨ R) (Premise) / ∴ ~T
4. ~S & ~R
5. ~S
6. ~P
7. ~R
8. ~P & ~R
9. ~(P ∨ R)
10. ~T

Exercise 9-12

Derive the indicated conclusions from the premises supplied.

▲ 1.
1. P & Q
2. P → R / ∴ R

▲ 2.
1. R → S
2. ~P ∨ R / ∴ P → S

3.
1. P ∨ Q
2. R & ~Q / ∴ P

▲ 4.
1. ~P ∨ (~Q ∨ R)
2. P / ∴ Q → R

5.
1. T ∨ P
2. P → S / ∴ ~T → S

6.
1. Q ∨ ~S
2. Q → P / ∴ S → P

7.
1. ~S ∨ ~R
2. P → (S & R) / ∴ ~P