Example 1-2. According to the relaxed definition, the error in our measurement of m_e is (9.15 - 9.11) x 10^{-31} kg = 0.04 x 10^{-31} kg.

Notice that the accepted value of m_e was rounded up to 9.11 x 10^{-31} kg because it would be pointless to use more significant figures in the accepted value than there are in the measured value.

There is a belief that quantities such as the mass of an electron have a true value and perfecting the measurement process would give better and better approximations to this true value. In contrast, there are many examples in which a true value is unknown because the value is not well defined. For example, the thickness of a block is unknown since even a very highly polished block has rough surfaces when viewed at high magnification. Therefore, the distance from one side is not well defined and improving the measuring instruments beyond some point would do no good.

2. Relative error is the error in a quantity divided by the quantity.

3. Uncertainty means the estimated reliability of a value. There are various ways to make this estimate and the main purpose of this first "experiment" is to gain skill in evaluating and expressing uncertainty. Thus, the figure 0.0000054 x 10^{-31} kg in the above-quoted mass of the electron is the uncertainty.

4. Relative uncertainty is the uncertainty in a value divided by the value. It may be expressed as a fractional decimal or as a percentage.

Exercise 1-1. Find the relative uncertainty in the mass of the electron.

5. Discrepancy means the difference between two measured values of a quantity. We shall measure the value of the acceleration due to gravity, g, several times during the semester. Suppose that we get g = 9.910 m/s^2 and 9.805 m/s^2 using two techniques. The discrepancy in the two values is 0.105 m/s^2. In the Appendix on p. 73, we find the accepted value of g = 9.797 m/s^2 for Northridge.

Exercise 1-2. What are the errors in the measured values g = 9.910 m/s^2 and 9.805 m/s^2? The relative errors?

6. Random errors describe the fluctuations in observations that differ from experiment to experiment. Random errors are characterized by the property that they are just as likely to occur in one direction as in another. They are sometimes called accidental errors or errors of observation. They tend to be small.

7. Systematic errors invariably have the same magnitude. For example, a steel tape measure calibrated for use at 20° C will yield lengths that are in error by the same amount every time when used at 30° C. Systematic errors often result from poorly calibrated equipment or from personal habits of the investigator. Sometimes we may compensate for these errors. For example, given the coefficient of thermal expansion of the measuring tape, we are

able to make appropriate adjustments to the observed lengths.

Exercise 1-3. Would we need to reduce or increase the lengths measured at 30° C? Explain.

8. Accuracy describes finding values that are near the true or accepted value. Accuracy is derived from having small systematic and random errors.

9. Precision means repeatability of the measurements. High precision is a result of small random errors; however, systematic errors could render a precise measurement inaccurate. Three measurements of g = 9.723, 9.722, and 9.724 m/s^2 are precise but inaccurate.

Example 1-3. Three golfers are practicing their approach shots. Their caddies keep track of the results and chart them as shown in Fig. 1-1.

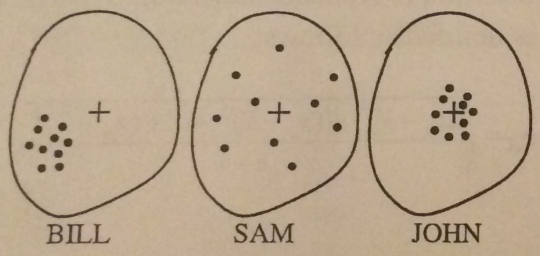


Fig. 1-1. Results of ten approach shots for Bill, Sam, and John respectively. The hole is marked by +.

The shots of Bill and John show high precision: they are reproduced. The accuracy of Bill's shots, however, is poor. The average position is short and to the left and is probably due to some systematic error in the conditions. It could be a personal error associated with an incorrect swing or an instrumental error if, for example, the clubs were slightly bent.

Turning to Sam's performance, it is noted that the average position of the balls is near the hole but they are widely scattered so they are not precise. Sam's performance is inaccurate statistically since any one shot is likely to be far from the hole. Obviously John's performance is the most desirable: accurate and precise.

Exercise 1-4. Suppose you measure the melting point of ice three times. Give an example of results that would be precise but inaccurate.

Significant figures

In recording observations it is always necessary to record all figures that can be trusted even though some of them are zeroes. As an example, suppose the length of a sheet of paper measured with an accuracy of 1 mm is found to be 28.0 cm. This is not to be recorded as 28 cm, because recording 28 cm implies that we have only determined that the length of the paper is nearer to 28 cm than to 27 or 29 cm. We have actually measured the length of the paper to three significant figures and our records should show that fact. On the other hand, if a man is weighed on a coarse scale capable of only 100-g increments, then it is incorrect to state his mass as 71300 g, because the two terminal zeroes were not measured. The mass is correctly recorded as 71.3 kg showing that the measurement was performed to three significant figures. Sometimes it is

necessary to use powers of 10 to record correctly the number of significant digits. For example, the speed of light measured to four significant figures is stated as 2.998×10^8 m/s.

Exercise 1-5. A pipe is measured once with a steel tape with markings every 0.1 mm. The two ends of the pipe line up with the 0 and 40-m marks on the tape. How should this datum be recorded using the appropriate number of significant figures?

10. Precision of an instrument is the least count or the smallest subdivision that may be read directly.

Treatment of random errors

Since random errors tend to compensate one another, it makes intuitive sense that the arithmetic mean $\bar{x}$ of n observations $x_1, x_2, \ldots, x_n$ defined as:

$$\bar{x} = \frac{x1 + x2 + \ldots + xn}{n} \qquad (1\text{-}1)$$

is a more reliable description of x than is any individual observation. The mean value is often called the average value.

Normally one more significant figure is retained for $\bar{x}$ than is appropriate for each individual x, because $\bar{x}$ is better defined than any individual measurement (see Beers' book).

The deviations of the values of the observations are defined to be $d1 = x1 - \bar{x}$, $d2 = x2 - \bar{x}$, ..., $dn = xn - \bar{x}$. It is easy to show that the sum of the deviations is zero but, of course, the sum of their squares is, in general, non-zero since each term is positive or zero.

Exercise 1-6. You measure the length of the table to be 182.4, 182.2, 182.5, 182.7, 182.4, 182.4, and 182.2 cm. What is the average length of the table? What are the deviations in each of the measurements? Add the deviations and show that you get zero.

Obviously, summing the deviations gives no indication of the scatter in the results of n measurements since you always get zero; however, adding the squares of the deviations is a measure of the scatter. If the observations are near one another, they will be close to $\bar{x}$ and all of the deviations and their squares will be small. A quantity related to the sum of the squares of the deviations is the sample standard deviation σ_x, which is defined as follows:

$$\sigma x = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n-1}} \qquad (1\text{-}2)$$

The numerator of the fraction under the square-root sign is just the sum of the squares of the deviations, so σ_X is one measure of the precision of the observations.

Exercise 1-7. Compute σ_x for the measurements in exercise 1-6.

Notice that these calculations can get tedious. Most calculators have a built-in

function to calculate eqs 1-1 and 1-2. It is well worth while to take the time to learn to use that function. Notice also the factor $n - 1$ in the denominator: there are functions on some calculators that compute eq 1-2 using a factor of n rather than $n - 1$; thus you need to check. Test your calculator's result against the long-hand calculation of exercise 1-7.

Histograms

When an experiment is repeated many times, it is customary to plot the data in a histogram.

Example 1-4. Consider the data of Table 1-1 detailing the results of 420 measurements of the diameter of a cylinder with an instrument of precision 0.1 mm. The histogram, using divisions of 0.1 mm, is shown in Fig. 1-2.

Table 1-1. Measurements of a cylinder

Diameter, mm	Frequency
20.4	2
20.5	12
20.6	10
20.7	33
20.8	40
20.9	76
21.0	80
21.1	65
21.2	52
21.3	22
21.4	12
21.5	5
21.6	11

The mean value could be computed from eq 1-1; however, there is no need to add 420 different numbers since we may combine all those of the same diameter. In general, denoting the frequency of x_n by f_n,

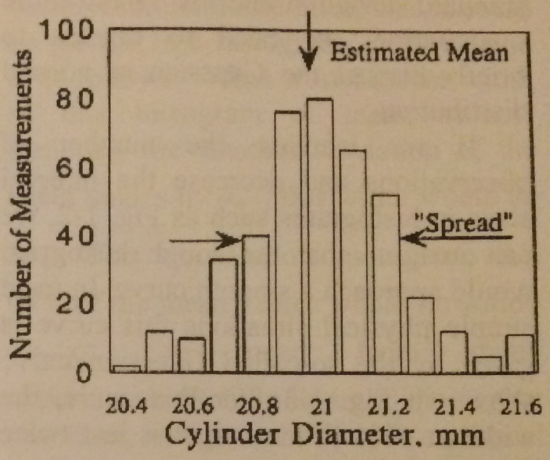


Fig. 1-2. Results of 420 measurements of the diameter of a cylinder.

$$\bar{x} = \frac{f1x1 + f2x2 + \ldots + fnxn}{f1 + f2 + \ldots + fn}, \quad (1\text{-}3)$$

and

$$\sigma_x = \sqrt{\frac{f1(x1-\bar{x})^2 + \ldots + fn(xn-\bar{x})^2}{f1 + \ldots + fn - 1}} \quad .(1\text{-}4)$$

Inserting the data of Table 1-1 into eqs 1-3 and 1-4 yields $\bar{x} = 21.00$ mm and $\sigma_x = 0.23$ mm.

Notice that the spread of the histogram in Fig. 1-2 gives an idea of the precision of the measurements. We have observed that the standard deviation is also a measure of the spread of results,

so the standard deviation and the histogram width must be related.

To make the connection between the standard deviation and the spread more quantitative, we need to digress to briefly discuss the Gaussian or normal distribution.

If we increase the number of observations and decrease the interval sizes in histograms such as Fig. 1-2, we can imagine that the rough histogram would approach a smooth curve. In most simple physical situations this curve is given by the so-called Gaussian curve shown in Fig. 1-3. For this curve, the width at 61% of the height is just twice the standard deviation.

Returning to the example of Fig. 1-2, let us roughly take the spread to be the combined width of the 5 central intervals, or 0.5 mm. For a perfect Gaussian curve, this width would be equal to twice the standard deviation, so estimating the standard deviation from the "spread" gives 0.5 mm/2 = 0.25 mm, which is very similar to the above computed 0.23 mm. For curves such as Fig. 1-3, the number of measurements in a given interval is given by the area under the curve for that interval. The area under the entire curve is the total number of measurements. The area under the curve between the limits of $+\sigma$ and $-\sigma$ of the mean turns out to be 68% of the total number of measurements. Another way to express this important result is as follows: There is a 68% probability that a measurement will fall within $\pm\sigma$ of the mean.

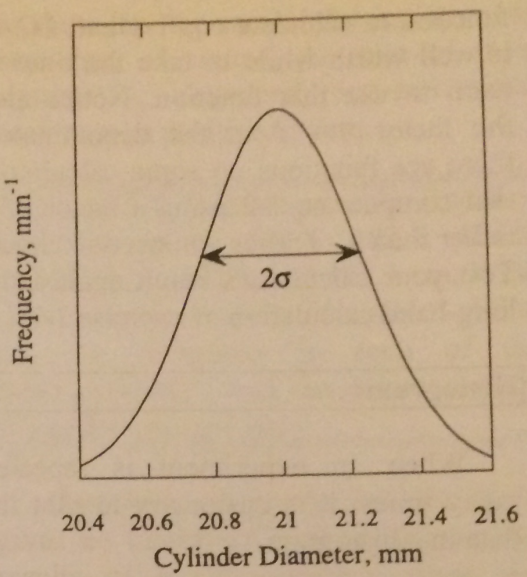


Fig. 1-3. The normal or Gaussian distribution that approximates the results in Fig. 1-2. The curve has width[1] of twice the standard deviation.

Exercise 1-8. Find the total number of measurements within the 5 central intervals of Fig. 1-2 by using the data of Table 1-1. Express your result as a percentage of the total number of measurements.

For exercise 1-8, if the histogram were a perfect Gaussian curve and the "spread" were exactly 2σ, the counted number of measurements would be 68% of the total number of measurements.

If we were to measure the cylinder once more, what would we get? Obviously, this is unknown; however, we are confident that the results will be near 21.0 mm. In fact, the result would

[1] This width is measured at 61% of the height of the curve.

be within ± 0.23 mm of 21.0 mm with a probability of 68%.

Now, we are in a position to complete the discussion of the arithmetic mean $\bar{x}$ and the sample standard deviation σ_x defined in eqs 1-1 and 1-2. The theory of error developed in the references shows:

(1) The arithmetic mean $\bar{x}$ is the best estimate available of the true value in the sense that it minimizes the sum of the squares of the deviations, and

(2) The sample deviation, eq 1-2, is the best estimate of the standard deviation of the Gaussian distribution.

The significance of $\bar{x}$ and σ_x

$\bar{x}$: The best available value from the data.

σ_x: A single measurement has a 68% probability of being within $\pm\sigma_x$ of $\bar{x}$.

Estimated error of the mean and its significance

We have learned that σ_X describes the probability that a single measurement will be within 68% of the mean. We are mainly interested in the uncertainty in the mean value. For the cylinder example, what is our confidence in the value of 21.00 mm? One way to answer this question would be to repeat the 420 measurements and get a new mean. Repeating this over and over again would produce many mean values and then these mean values could be plotted in a histogram. Our question is as follows: What would be the width of this histogram of mean values? Denoting the standard deviation of the mean values by $\sigma_{\bar{x}}$, this width would be $2\sigma_{\bar{x}}$. This would allow us to state that 68% of the mean values would lie within $\pm\sigma_{\bar{x}}$ of the mean from many experiments. A little thought shows that one does not expect different statistical deviations in a second set of 420 measurements and the spread of mean values ought to be available from just one set of measurements. The references show this to be so and that the standard deviation of the mean values is:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}. \qquad (1\text{-}5)$$

The significance of σ_x and $\sigma_{\bar{x}}$

σ_x : A single measurement has a 68% probability of being within $\pm\sigma_X$ of $\bar{x}$.

$\sigma_{\bar{x}}$: The mean value has a 68% probability of being within $\pm\sigma_{\bar{x}}$ of $\bar{x}$.

The standard deviation of the mean is also called the estimated error of the mean.

Estimated uncertainty

Sometimes we make several measurements that are identical. Eq 1-2 would yield $\sigma_X = 0$, which could lead to the mistaken impression that the uncertainty is zero when in fact the precision of the instrument is less than the random errors. We must estimate the uncertainty, and, in most cases, the precision of the instrument is used as the uncertainty.

Sometimes only one measurement is made and often the precision of the instrument is used as an estimate of the uncertainty. Depending upon the instrument and the situation, the uncertainty could be less than the precision of the instrument or it could be more. For example, in measuring the length of a sheet of paper with a meter stick, we may estimate the distance between the smallest subdivision (mm) if we turn the meter stick onto its side to eliminate parallax. Therefore, the uncertainty in the measurement might be 0.5 mm or even 0.3 mm rather than the precision of the meter stick which is 1 mm. On the other hand, if we measure an object that is not right next to the meter stick, the uncertainty might be 2 or 3 mm. Often judgment is required in order to estimate the uncertainty in a single measurement.

We shall employ the same symbol, σ_X, to denote estimated uncertainty in this manual.

Reporting uncertainties

The uncertainty in an experimental quantity must be reported. Since there are a number of ways to do this we must state the meaning of the uncertainty. A common method is to report $\bar{x} \pm \sigma_x$ together with the number of measurements n. The reader can quickly calculate $\sigma_{\bar{x}}$. In the remainder of this manual, we shall either quote the standard deviation or the estimated uncertainty as the uncertainty.

Table 1-2. Weight of a flea.

Measurement	Weight, N
1	$W_1 = 1.96 \times 10^{-4}$
2	$W_2 = 1.94 \times 10^{-4}$
3	$W_3 = 2.01 \times 10^{-4}$
4	$W_4 = 2.00 \times 10^{-4}$
5	$W_5 = 1.97 \times 10^{-4}$
6	$W_6 = 1.99 \times 10^{-4}$
7	$W_7 = 2.03 \times 10^{-4}$
8	$W_8 = 1.98 \times 10^{-4}$
9	$W_9 = 1.96 \times 10^{-4}$
10	$W_{10} = 1.95 \times 10^{-4}$
11	$W_{11} = 2.00 \times 10^{-4}$
12	$W_{12} = 1.97 \times 10^{-4}$

Example 1-5: To illustrate these ideas, we might measure the weight of a flea twelve times with a balance of precision 0.01×10^{-4} N. The results are tabulated in Table 1-2.

The mean weight, according to eq 1-1 is:

$$W = \frac{W_1 + W_2 + \ldots + W_n}{n}$$

$$= \frac{1.96 + 1.94 + ... + 1.97}{12} \times 10^{-4} \text{ N}$$

$$= 1.980 \times 10^{-4} \text{ N}.$$

Note that we have retained one more significant figure for W , for the purpose of calculating the sample standard deviation which is

$$\sqrt{\frac{(1.96-1.980)^2+...+(1.97-1.980)^2}{12 - 1}} \times 10^{-4} \text{ N}$$

$$\sigma_W = 0.02662874 \times 10^{-4} \text{ N}.$$

An uncertainty is really only relevant to one digit; if the "0.02" represents uncertainty, the following "6" in "0.026" is even more so. A common practice is to keep two digits and quote the results of the measurements as $W = (1.980 \pm 0.027) \times 10^{-4}$ N although $W = (1.98 \pm 0.03) \times 10^{-4}$ N is correct. The interval of $\pm\sigma_W$ about W includes all values between 2.007 and 1.953 x 10^{-4} N. Checking Table 1-2, we see that 8 out of the 12 values, or 67%, fall within this spread, which is near the known result for a Gaussian in which case 68% of the measurements fall within $\pm\sigma_X$ of the mean.

Turning to the estimated error of the mean, from eq 1-5

$$\sigma_{\bar{W}} = \frac{\sigma_W}{\sqrt{12}} = 0.0077 \times 10^{-4} \text{ N}.$$

Thus our mean value of W = 1.980 x 10^{-4} N has a 68% chance of being within 0.008 x 10^{-4} N of a mean computed from many experiments.

Exercise 1-9. Suppose that the flea weighing experiment had been terminated after 8 weightings. Find $\bar{W}$ and σ_W. Count how many values fall within $\bar{W} \pm \sigma_W$ and compare with the expected result from a Gaussian distribution.

Exercise 1-10. Table 1-3 shows the results of tossing 10 pennies at once 84 different times. Draw a histogram to depict the results. On your histogram draw arrows to indicate your estimate of the mean value of the number of heads and the spread (See Fig. 1-2). Write down the estimated values of the mean number of heads and the standard deviation.

Table 1-3. Tossing 10 pennies.

Number of Heads	Frequency
0	2
1	1
2	9
3	14
4	16
5	19
6	12
7	8
8	2
9	1
10	0

Exercise 1-11. Find the mean number of heads and the sample standard deviation in Table 1-3 using eqs 1-3 and 1-4. Compare your answers with your estimates in exercise 1-10.

Propagation of errors

Practically all physical quantities of interest are calculated with a formula. For example, to measure a speed, *v*, we actually measure a distance, *d*, and the elapsed time, *t*, and compute v=d/t. Both *d* and *t* will have uncertainties. As a second example, if we want the area of a sheet of paper we would measure the length *L* and width *W* and calculate A = LW. To calculate either *v* or *A*, we would use mean values of *d* and *t* or *L* and *W*; however, we drop the bars indicating mean values for convenience. In these two examples, what are the uncertainties in the calculated quantities *v* and *A*?

Example 1-6: Suppose a sheet of paper is measured to have dimensions L= 25.0 ± 1.0 cm and W = 15.0 ± 1.0 cm. Therefore *A*= 375 cm^2. Taking the extremes, *L* could be as large as 26.0 cm and *W* as large as 16.0 cm giving *A* = 416 cm^2; or L could be as small as 24.0 cm and *W* as small as 14.0 cm, giving *A* = 336 cm^2. Thus, a worst case estimate of the spread is 416 cm^2 - 336 cm^2 = 80 $cm^2 = 2\sigma_A$, thus σ_A = 40 cm^2. We call it a "worst case" estimate because it is unlikely that deviations in both *L* and *W* would be at one extreme or the other. The relative uncertainty $\frac{\sigma_A}{A}$ = 40 cm^2/375 cm^2 = 0.107 for this worst case estimate.

By taking into account that deviations in *W* and *L* are uncorrelated – i.e., sometimes they are both positive, both negative, or of opposite sign – the references show that for normal distributions

$$\frac{\sigma_A^2}{A^2} = \frac{\sigma_L^2}{L^2} + \frac{\sigma_W^2}{W^2}. \qquad (1\text{-}7)$$

Using the numbers from example 1-6, $\frac{\sigma_A}{A}$ = 0.078 by eq 1-7. Comparing the relative uncertainty calculated correctly using eq 1-7 with the worst case calculation, we see that the worst case analysis overestimates the error in *A* by some 37%.

How about the uncertainty in the speed calculated from *v=d/t*? In other words, how are uncertainties propagated by division? Curiously, the answer is the same for division as it was for multiplication:

$$\frac{\sigma_v^2}{v^2} = \frac{\sigma_d^2}{d^2} + \frac{\sigma_t^2}{t^2}. \qquad (1\text{-}8)$$

Many calculations, including simple multiplication and division, may be written as a product of terms with each term raised to a power:

$$Z = a\,x^b\,y^c\,w^d\,. \qquad (1\text{-}9)$$

The equation *A = LW* may be written in the form of eq 1-9 with *Z = A*, *a* = 1, *x* = L, *b* = 1, *y = W*, *c* = 1, *w* = 1, and *d* = 1. The computation *v = d/t* may be written in the same form with *Z = v*, *a* = 1, *x = d*, *b* = 1, *y = t*, *c* = -1, *w* = 1, and *d* = 1.

For equations of the form of eq 1-9, the propagated uncertainty is given by

$$\frac{\sigma_Z^2}{Z^2} = b^2\frac{\sigma_x^2}{x^2} + c^2\frac{\sigma_y^2}{y^2} + d^2\frac{\sigma_w^2}{w^2}\,. \qquad (1\text{-}10)$$

Applied to either simple division or multiplication, eq 1-10 yields eqs 1-7 and 1-8 respectively.

Example 1-7: We measure the length of a cylinder to be $L = 5.63 \pm 0.02$ cm and the diameter to be $D = 2.81 \pm 0.03$ cm and wish to find the volume.

$$V = \frac{\pi D^2 L}{4} = 34.9149091 \text{ cm}^3.$$

Obviously, we have kept too many significant figures. Now V is of the form of eq 1-9 with $Z = V$, $a = \pi/4$, $x = D$, $b = 2$, $y = L$, $c = 1$, $w = 1$, and $d = 1$. We have from eq 1-10:

$$\frac{\sigma_V^2}{V^2} = 4\,\frac{\sigma_D^2}{D^2} + \frac{\sigma_L^2}{L^2}$$

$$= 4\,\frac{(0.03 \text{ cm})^2}{(2.81 \text{ cm})^2} + \frac{(0.02 \text{ cm})^2}{(5.63 \text{ cm})^2} = 0.000469.$$

Solving for σV gives $\sigma V = 0.76$ cm^3. We cannot trust the value of V beyond $\pm$ 0.8 cm^3, so we round to tenths of cm^3 and write the following:

$$V = (34.9 \pm 0.8) \text{ cm}^3,$$

or, if you wish,

$$V = (34.91 \pm 0.78) \text{ cm}^3,$$

Example 1-8: The period of a satellite is given by Kepler's third law as:

$$T = \sqrt{\frac{4\pi^2 m a^3}{K}}$$

where K is a constant, m is the reduced mass, and a is the semi-major axis of the elliptical orbit. We wish to compute the standard deviation in T given the standard deviation in m, a, and K. Write T in the form of eq 1-9.

$$T = 2\pi m^{1/2}\, a^{3/2}\, K^{-1/2}.$$

Utilizing eq 1-10,

$$\frac{\sigma_T^2}{T^2} = \frac{1}{4}\frac{\sigma_m^2}{m^2} + \frac{9}{4}\frac{\sigma_a^2}{a^2} + \frac{1}{4}\frac{\sigma_K^2}{K^2}.$$

Exercise 1-12. A box is measured to be of length $L = 10.13 \pm 0.01$ ft, width $W = 1.534 \pm 0.002$ ft and height $H = 0.013 \pm 0.001$ ft. Calculate the volume. Write the expression for V in the form of eq 1-9 and find the expression for σV. Compute σV.

Exercise 1-13. The length of a simple pendulum is $L = 133.15 \pm 0.06$ cm. The acceleration due to gravity is known to be $g = 979.20 \pm 0.04$ cm/s^2. Calculate the period of the pendulum from $T = 2\pi\sqrt{L/g}$. Write the expression for T in the form of eq 1-9 and find the expression for σT. Compute σT.

Example 1-9: Suppose we wish to calculate the perimeter of the sheet of paper in example 1-6. The equation for the perimeter, P, is:

$$P = 2L + 2W. \qquad (1\text{-}11)$$

Thus, $P = 2(25 \text{ cm}) + 2(15 \text{ cm}) = 80$ cm. Equation 1-11 is not of the form of eq 1-9, so eq 1-10 cannot be used to calculate the uncertainty.

Equations involving addition may be written in the general form:

$$Z = ax + by + cw\,. \qquad (1\text{-}12)$$

The references show that the uncertainty may be computed from:

$$\sigma_Z^2 = a^2\sigma_x^2 + b^2\sigma_y^2 + c^2\sigma_w^2\,. \qquad (1\text{-}13)$$

Applying eq 1-13 to eq 1-11 yields:

$$\sigma_P^2 = 4\sigma_W^2 + 4\sigma_L^2$$
$$= 4(1\text{ cm})^2 + 4(1\text{ cm})^2.$$

So, $\sigma P = 2.8$ cm, thus we write:

$$P = 80 \pm 3 \text{ cm}.$$

Exercise 1-14. Calculate the worst case uncertainty in P as follows: first calculate one extreme of P by using L = 26 cm and W = 16 cm. Then calculate the other extreme using L = 24 cm and W = 14 cm. The worst case spread is the difference in these two extreme values of P. Set this spread equal to $2\ \sigma\ p$ and calculate $\sigma\ p$.

A general expression for σZ

Equations 1-9 and 1-13 cover many, but by no means all, expressions encountered in the physical sciences. In the more general case when Z is an arbitrary function of x, y, w, etc.

$$\sigma_z^2 = \left(\frac{\partial Z}{\partial x}\right)^2 \partial_x^2 + \left(\frac{\partial Z}{\partial y}\right)^2 \partial_y^2$$
$$+ \left(\frac{\partial Z}{\partial w}\right)^2 \partial_w^2 \qquad (1\text{-}14)$$

The symbol $\frac{\partial z}{\partial x}$ is the partial derivative of Z with respect to x which is found by taking the derivative of Z with respect to x while holding y and w constant.