COMPLEX NUMBERS

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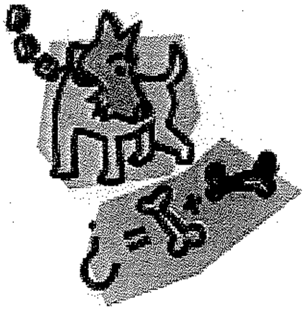


COMPLEX NUMBERS.

1 Why study complex numbers ?

Mathematics:

1. complex numbers make many parts of calculus much easier - particularly differential equations and integrals
2. complex numbers can be used for many (but not all) purposes where 2D vectors are used
3. complex numbers show that the functions $\sin x$, $\cos x$ and e^x are basically the same thing
4. complex numbers are needed to solve the general quadratic equation $ax^2 + bx + c = 0$ where a, b, c are real constants.

Physics:

1. description of wave motion is greatly simplified using complex numbers
2. Schrodinger matter waves are intrinsically complex mathematical quantities

Engineering:

1. analysing AC electrical circuits
2. signal processing
3. analysing vibrations of structures

2 Definition of i.

The "square root of negative one" is written i , and satisfies

$$i^2 = -1 \tag{2.1}$$

so that in some sense $i = \sqrt{-1}$.

Clearly no real number has the property (2.1), since the square of any real number x (whether x is positive, zero or negative) cannot be negative.

Thus i is a non-real number, known as an **imaginary number**.
This is a class of numbers you may not have met before.

Apart from Eq. (2.1), all the other properties of i are determined by demanding that

i is a number obeying all the laws of algebra.

Notes:

1. Do not confuse the imaginary number i with the unit vector $\underset{\sim}{\hat{i}}$.
2. Some engineering texts use the symbol j for $\sqrt{-1}$.
   Do not confuse this with the unit vector $\underset{\sim}{\hat{j}}$.

3 Imaginary Numbers.

Imaginary numbers are made by multiplying i by a real number.

Thus the following numbers are imaginary :

$3i$

$-(2.5)i$

$i\pi$

$\frac{3i}{4}$ or $\frac{3}{4}i$

$i\ (= 1 \times i)$

$-i\ (= -1 \times i)$.

If a is a positive real number, then – a is a real negative number and therefore

$$\sqrt{-a} = \sqrt{-1 \times a} = \pm\ \sqrt{-1} \times \sqrt{a} = \pm\ i\sqrt{a}.$$

We can check this by squaring either $i\sqrt{a}$ or $-i\sqrt{a}$ and see that we get – a.

$$\left(i\sqrt{a}\right)^2 = i\sqrt{a} \times i\sqrt{a} = i^2\left(\sqrt{a}\right)^2 = (-1) \times a = -a$$

or $$\left(-i\sqrt{a}\right)^2 = \left(-i\sqrt{a}\right) \times \left(-i\sqrt{a}\right) = (-i)^2\left(\sqrt{a}\right)^2 = i^2\left(\sqrt{a}\right)^2 = (-1) \times a = -a.$$

This is no different to real numbers as all real numbers have two square roots differing by a factor of –1, that is $\sqrt{a^2} = \pm\ a$.

Examples:

1. $\sqrt{-1} = i$ and $-i$.

2. $\sqrt{-4} = 2i$ and $-2i$.

3. $\sqrt{-5} = i\sqrt{5}$ and $-i\sqrt{5}$.

4 Complex Numbers.

A **complex number** is defined as the sum of a real and an imaginary number.

A general complex number is often denoted by the symbol z. Thus

- any complex number z can be written as $\mathbf{z = \underset{\text{real part}}{\underset{\uparrow}{x}} + i\,\underset{\text{imaginary part}}{\underset{\uparrow}{y}}}$ where x and y are real numbers

- x is referred to as the **real** part of z and is written as $\mathbf{x = Re(z)}$
- y is referred to as the **imaginary** part of z and is written as $\mathbf{y = Im(z)}$.

*Note: The imaginary part of z, that is Im(z) is y **not** i y.*

Examples of complex numbers:

$5 + 2i$

$7 - 3i = 7 + (-3)i$

$-\pi + 3.78i$

$-\frac{2}{3} + \frac{9}{10}i$

$\sqrt{3} - \sqrt{5}\,i$

are all complex numbers.

The real numbers such as $6 = 6 + 0i$

and the imaginary numbers such as $5i = 0 + 5i$

are special cases of complex numbers.

Complex numbers can be added, subtracted, multiplied and divided using the usual laws of algebra, with the addition of the definition $i^2 = -1$.

5 The Arithmetic of Complex Numbers.

As a general rule you should simplify any complex number to $x + iy$.

Adding complex numbers.

Consider two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= x_1 + iy_1 + x_2 + iy_2 && \textit{expanding out the brackets} \\ &= x_1 + x_2 + iy_1 + iy_2 && \textit{rearranging the terms} \\ &= \underbrace{(x_1 + x_2)}_{\text{real part}} + \underbrace{i(y_1 + y_2)}_{\text{imaginary part}} && \textit{collecting real and imaginary terms together} \end{aligned}$$

Thus when we add two complex numbers together, we simply add the two real parts and then add the two imaginary parts.

Example:

$(3 + 5i) + (2 - 4i) = (3 + 2) + (5 - 4)i = \mathbf{5 + i}$.

Subtracting complex numbers.

Using the complex numbers z_1 and z_2 from above, we have

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= x_1 + iy_1 - x_2 - iy_2 && \textit{expanding out the brackets} \\ &= x_1 - x_2 + iy_1 - iy_2 && \textit{rearranging the terms} \\ &= \underbrace{(x_1 - x_2)}_{\text{real part}} + \underbrace{i(y_1 - y_2)}_{\text{imaginary part}} && \textit{collecting real and imaginary terms together} \end{aligned}$$

Thus when we subtract one complex number from another, we simply subtract the real parts and then subtract the imaginary parts.

Example:

$(3+5i) - (2-4i) = (3-2) + [5-(-4)]i = (3-2) + (5+4)i = \mathbf{1+9i}.$

Multiplying complex numbers.

Now consider multiplying z_1 and z_2:

$$\begin{aligned} z_1 \times z_2 &= (x_1+iy_1)(x_2+iy_2) \\ &= x_1x_2 + x_1(iy_2) + (iy_1)x_2 + (iy_1)(iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + (-1)y_1y_2 \qquad \textit{using } i^2 = -1 \\ &= \underbrace{\mathbf{(x_1x_2 - y_1y_2)}}_{\text{real part}} + \mathbf{i}\underbrace{\mathbf{(x_1y_2 + x_2y_1)}}_{\text{imaginary part}}. \end{aligned}$$

Do not try to memorize this formula as it is easy to just repeat the lines above.

Example:

$$\begin{aligned} (3+5i)(2-4i) &= 3\times2 + 3(-4i) + (5i)2 + (5i)(-4i) \\ &= 6 - 12i + 10i + 5(-4)i^2 \\ &= 6 - 12i + 10i + 5(-4)(-1) \\ &= 6 + (-12+10)i + 20 \\ &= (6+20) + (-12+10)i \\ &= \mathbf{26 - 2i}. \end{aligned}$$

With practice, you may be able to skip some of the steps above.

Dividing complex numbers.

This also uses ordinary algebra, but the answer is simplified by a special trick. Therefore we will look at dividing complex numbers a little later.

Equality of complex numbers.

Two complex numbers are equal if and only if they have the same real part and the same imaginary part.

In terms of symbols,

Consider the two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

$z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

6 The Argand Plane.

Geometric representation of complex numbers.

A complex number $z = x + iy$ is often represented by the point (x, y) in a two-dimensional ("2D") diagram with axes, just as in plotting a point on a graph (Figure 1).

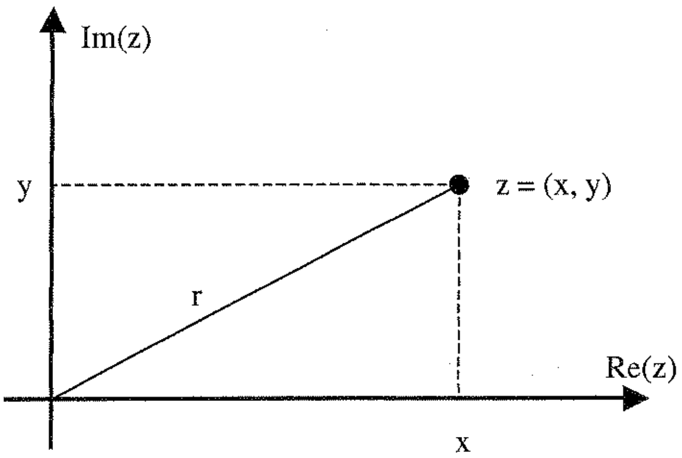


Figure 1 : Argand diagram.

This diagram is called an **Argand diagram** and the x – y plane in which it is drawn is known as the **complex plane** or sometimes the **Argand plane**. The point z is here represented by its Cartesian coordinates x and y.

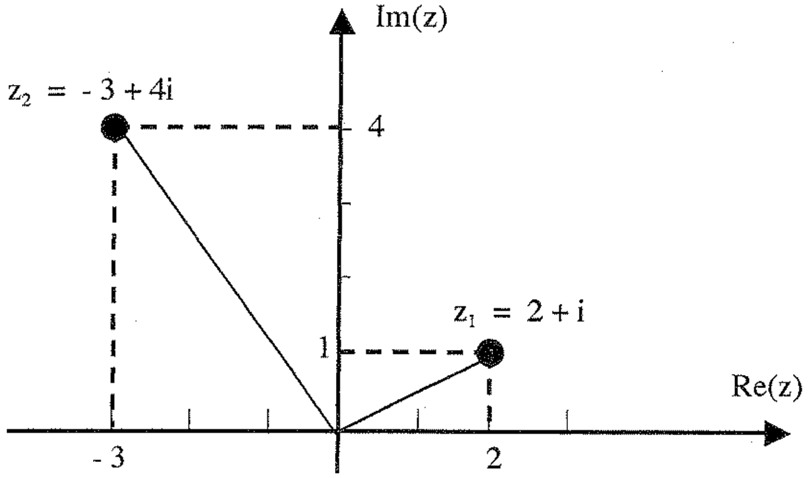
The form $\mathbf{z = x + iy}$ is called the **Cartesian form of the complex number z.**

The horizontal or Re(z) axis is termed the **real axis**.

The vertical or Im(z) axis is termed the **imaginary axis**.

Example :

Draw the complex numbers $z_1 = 2 + i$ and $z_2 = -3 + 4i$ on an Argand diagram.



7 Modulus of a complex number.

The **modulus**, r, of a complex number, z, is denoted by $|z|$.

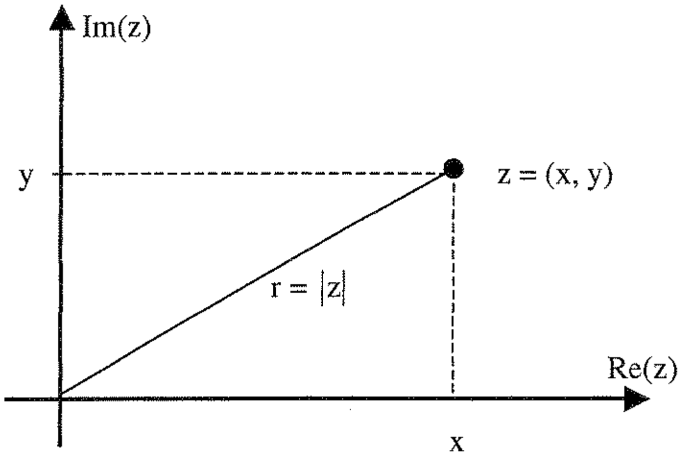


Figure 1 : Argand diagram.

Using Pythagoras' Theorem in the triangle of Figure 1, we find that

$$|z| = |x + iy| = \sqrt{x^2 + y^2}. \tag{7.1}$$

The modulus of the complex number z, $|z|$, is also sometimes called the **absolute value** of z, the **magnitude** of z or the **length** of z.

Example:

The modulus of $3 + 2i$ is $|3 + 2i| = \sqrt{3^2 + 2^2} = \sqrt{\mathbf{13}}$.

Exercise:

For the special case of a real number $x = x + 0i$, check that our definition of absolute value $|x + 0i|$ agrees with the definition of the absolute value $|x|$ already encountered in calculus.

Examples:

If $p = 2 - 7i$ and $q = -3 - 5i$, find

- **(a)** $2p + q$
- **(b)** $3p - 2q$
- **(c)** pq
- **(d)** $\mathrm{Re}(2p + q)$
- **(e)** $\mathrm{Im}(3p - 2q)$
- **(f)** $|p|$
- **(g)** Draw p and q on an Argand diagram.

(a) $2p + q = 2(2 - 7i) + (-3 - 5i) = 4 - 14i - 3 - 5i$

$= (4 - 3) + (-14 - 5)i = \mathbf{1 - 19i}.$

(b) $3p - 2q = 3(2 - 7i) - 2(-3 - 5i) = 6 - 21i + 6 + 10i$

$= (6 + 6) + (-21 + 10)i = \mathbf{12 - 11i}.$

(c) $pq = (2 - 7i)(-3 - 5i) = 2(-3) + 2(-5i) + (-7i)(-3) + (-7i)(-5i)$

$= -6 - 10i + 21i + 35i^2 = -6 - 10i + 21i + 35(-1)$

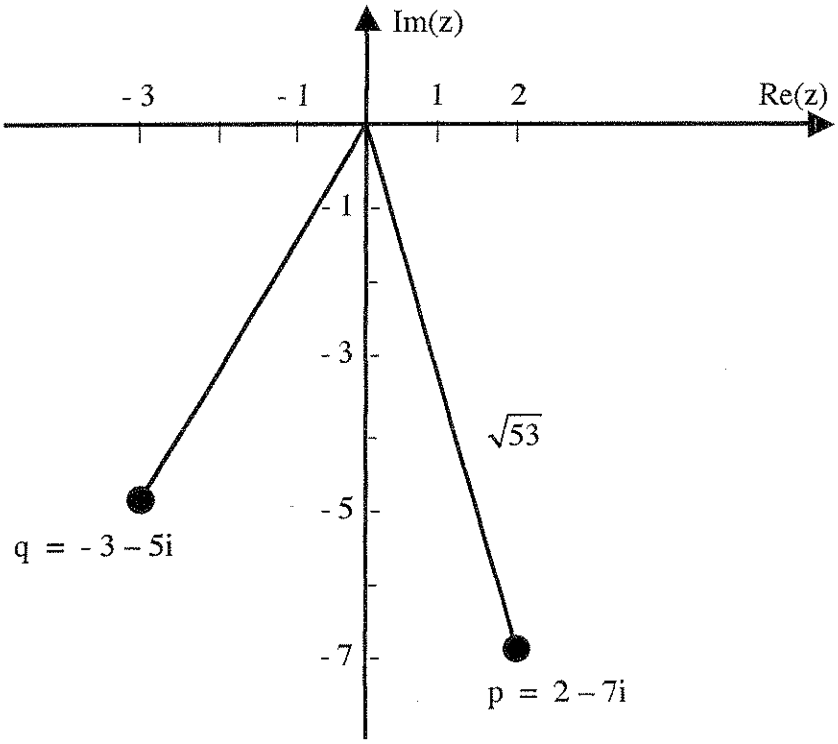
$= (-6 - 35) + (-10 + 21)i = \mathbf{-41 + 11i}.$

(d) $\mathrm{Re}(2p + q) = \mathrm{Re}(1 - 19i) = \mathbf{1}.$

(e) $\mathrm{Im}(3p - 2q) = \mathrm{Im}(12 - 11i) = \mathbf{-11}.$

(f) $|p| = |2 - 7i| = \sqrt{2^2 + (-7)^2} = \sqrt{4 + 49} = \sqrt{\mathbf{53}}.$

(g)



8 The complex conjugate, $\bar{z}$.

The complex conjugate, $\bar{z}$, of a complex number $z = x + iy$ is defined as the complex number having the same real part but imaginary part of opposite sign, thus

$$\bar{z} = \overline{x + iy} = x - iy \quad \text{where x and y are real.}$$

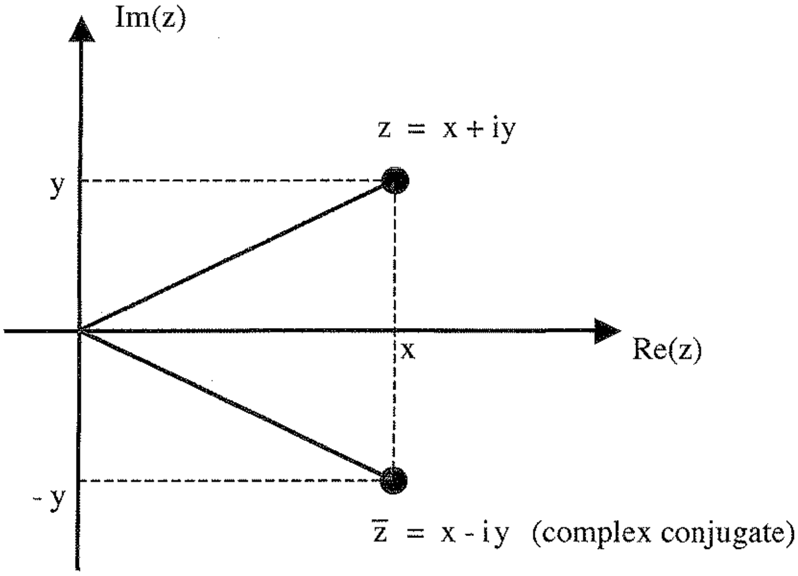


***Figure 2:** Complex conjugate $\bar{z}$.*

This is illustrated in Figure 2. We can think of the complex conjugate of z as the reflection of z in the real (horizontal) axis.

Example:

Let $z = 2 + 5i$, then $\bar{z} = \overline{2 + 5i} = 2 - 5i$.

The complex conjugate has the following useful property.

Multiplying an arbitrary complex number $z = x + iy$ by its conjugate $\bar{z} = x - iy$ gives

$$z\bar{z} = (x + iy)(x - iy) = x^2 + x(-iy) + (iy)x + (iy)(-iy)$$

$$= x^2 - ixy + ixy - i^2y^2 = x^2 - ixy + ixy - (-1)y^2 \quad (\textit{using } i^2 = -1)$$

$$= \mathbf{x^2 + y^2}.$$

From Equation (7.1) we find that $x^2 + y^2 = |z|^2$ so that

$$z\overline{z} = |z|^2 = x^2 + y^2.$$

The process of multiplying by the complex conjugate is called ***rationising z.***

Since $x^2 + y^2 = |z|^2$ is a real non-negative number, rationalisation converts a complex number to a real number.

Note that by addition and subtraction of $z = x + iy$ and its complex conjugate, $\overline{z} = x - iy$ we get

$$z + \overline{z} = 2x \qquad \text{and} \qquad z - \overline{z} = 2iy$$

which leads to the two important formulae

$$\mathrm{Re}(z) = x = \frac{1}{2}(z + \overline{z})$$

$$\mathrm{Im}(z) = y = \frac{1}{2i}(z - \overline{z}).$$

Example:

Let $z = 4 + 3i$ then

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}\left[(4 + 3i) + (4 - 3i)\right] = \frac{4 + 4}{2} = \frac{8}{2} = 4$$

$$\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z}) = \frac{1}{2i}\left[(4 + 3i) - (4 - 3i)\right] = \frac{3i + 3i}{2i} = \frac{6i}{2i} = 3.$$

Properties of Complex Conjugates.

For the two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have the following

$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$$

$$\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$$

$$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2 .$$

Let us prove the first result.

We need to show that the left hand side (LHS) equals the right hand side (RHS).

Firstly, expand the LHS

$$\begin{aligned} \text{LHS} &= \overline{z_1 + z_2} \\ &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\ &= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

Next, expand the RHS

$$\begin{aligned} \text{RHS} &= \overline{z}_1 + \overline{z}_2 \\ &= \overline{x_1 + iy_1} + \overline{x_2 + iy_2} \\ &= (x_1 - iy_1) + (x_2 - iy_2) \\ &= (x_1 + x_2) - i(y_1 + y_2) \qquad \textit{(collecting real and imaginary parts)} \\ &= \text{LHS} \end{aligned}$$

Since the RHS equals the LHS, we have proved the result.

The proofs of the remaining two results are left as an exercise.

Example:

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$, then

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(4 + 3i)(2 + 5i)} \\ &= \overline{[4 \times 2 + 4(5i) + (3i)2 + (3i)(5i)]} \\ &= \overline{8 + 20i + 6i - 15} \\ &= \overline{-7 + 26i} = \mathbf{-7 - 26i}. \end{aligned}$$

9 Dividing complex numbers in Cartesian form.

To divide two complex numbers z and w to form the complex number $\frac{z}{w}$, we multiply top and bottom by the complex conjugate of the bottom line, $\bar{w}$.

This process of multiplying top and bottom does not change the value (after all we have only multiplied by $\frac{\bar{w}}{\bar{w}} = 1$) but it converts the denominator to a real number $w\bar{w} = |w|^2$, and division by a real number is easy.

Example:

If $z_1 = 3 + 5i$ and $z_2 = 2 - 4i$, find $\frac{z_1}{z_2}$.

As we are dividing by z_2, we need to find the complex conjugate of z_2.

Therefore $\quad \bar{z}_2 = \overline{2 - 4i} = 2 + 4i$.

$$\frac{z_1}{z_2} = \frac{3+5i}{2-4i} = \frac{3+5i}{2-4i} \times \frac{2+4i}{2+4i} \qquad \textit{(multiply top \& bottom by conjugate of } z_2\textit{)}$$

$$= \frac{(3+5i)(2+4i)}{(2-4i)(2+4i)}$$

$$= \frac{3\times 2 + 3(4i) + (5i)2 + (5i)(4i)}{2\times 2 + 2(4i) + (-4i)2 + (-4i)(4i)} \qquad \textit{(expand out top and bottom)}$$

$$= \frac{6 + 12i + 10i + 20i^2}{4 + 8i - 8i - 16i^2} = \frac{6 + 12i + 10i - 20}{4 + 8i - 8i + 16} \qquad \textit{(simplify using } i^2 = -1\textit{)}$$

$$= \frac{6 - 20 + (12 + 10)i}{4 + 16 + (8 - 8)i} \qquad \textit{(collect real \& imaginary parts together)}$$

$$= \frac{-14 + 22i}{20 + 0i} = -\frac{14}{20} + \frac{22}{20}i \qquad \textit{(simplify)}$$

this must be zero

$$= -\frac{\mathbf{7}}{\mathbf{10}} + \frac{\mathbf{11}}{\mathbf{10}}\mathbf{i}.$$

Check:

We claim to have found $p = \frac{z_1}{z_2}$ where $z_1 = 3 + 5i$ and $z_2 = 2 - 4i$.

If this is true, we should be able to find that $pz_2 = z_1$.

We can do this my multiplying p by z_2 and show that it is equal to z_1.

$$\begin{aligned}
pz_2 &= \left(-\frac{7}{10} + \frac{11}{10}i\right)(2 - 4i) \\
&= \left(-\frac{7}{10}\right)2 + \left(-\frac{7}{10}\right)(-4i) + \left(\frac{11}{10}i\right)2 + \left(\frac{11}{10}i\right)(-4i) \\
&= -\frac{14}{10} + \frac{28}{10}i + \frac{22}{10}i - \frac{44}{10}i^2 \\
&= -\frac{14}{10} + \frac{28}{10}i + \frac{22}{10}i + \frac{44}{10} \qquad \textit{(using } i^2 = -1\textit{)} \\
&= \left(-\frac{14}{10} + \frac{44}{10}\right) + \left(\frac{28}{10} + \frac{22}{10}\right)i \qquad \textit{(collect real \& imaginary parts)} \\
&= \frac{30}{10} + \frac{50}{10}i = \mathbf{3 + 5i = z_1}.
\end{aligned}$$

10 Polar form of a complex number.

Two real numbers, x and y, are required to determine a complex number $z = x + iy$. This is termed the **Cartesian form of z.** But the same point (x, y) in the Argand plane could be specified by the **polar coordinates** – the "modulus" $r = |z|$ and the "argument", θ, as shown in Figure 3.

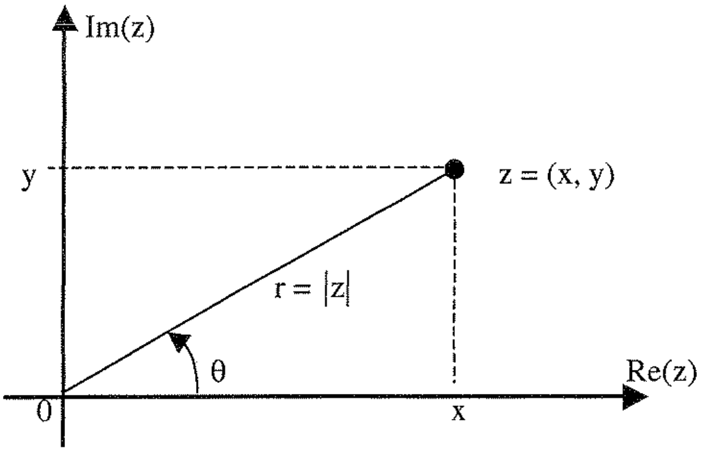


Figure 3 : Polar and Cartesian forms of z.

We write $\theta = \arg(z)$.

Note that the argument, θ, is sometimes called the "phase angle".

The form (r, θ) is termed the **polar form of z.**

(Later we will find a more useful representation of the polar form,
namely $r \exp(i\theta)$ or $r\,e^{i\theta}$).

Note that the argument θ is measured anticlockwise from the real axis.

A negative angle implies clockwise rotation from the real axis.

Ambiguity of the argument θ.

The physical or geometric angle shown in Figure 3 is about $45^\circ = \frac{\pi}{4}$ radians.

But the same **geometric** angle (i.e. θ in figure 3) would be obtained by rotating through an arbitrary number of complete circles. Thus, in general, the same diagram would be obtained with a numerical angle $\theta + 360n$ (in degrees) or $\theta + 2\pi n$ (in radians). Here n is an arbitrary integer. It could be 0, + 1, - 1, + 2, - 2, etc..
Thus in the particular case depicted in figure 3, the same geometric angle θ could be written numerically in degrees as

$$n = 0: \quad 45^\circ + (0)(360^\circ) = 45^\circ$$

$$n = 1: \quad 45^\circ + (1)(360^\circ) = 405^\circ$$

$$n = -1: \quad 45^\circ + (-1)(360^\circ) = -315^\circ$$

$$n = 2: \quad 45^\circ + (2)(360^\circ) = 765^\circ$$

$$n = -2: \quad 45^\circ + (-2)(360^\circ) = -675^\circ$$

In radians, the same geometric angle is given by the numerical angles

$$n = 0: \quad \frac{\pi}{4} + (0)(2\pi) = \frac{\pi}{4}$$

$$n = 1: \quad \frac{\pi}{4} + (1)(2\pi) = \frac{9\pi}{4}$$

$$n = -1: \quad \frac{\pi}{4} + (-1)(2\pi) = -\frac{7\pi}{4}$$

$$n = 2: \quad \frac{\pi}{4} + (2)(2\pi) = \frac{17\pi}{4}$$

$$n = -2: \quad \frac{\pi}{4} + (-2)(2\pi) = -\frac{15\pi}{4}$$

It is customary (but not universal) to choose n so that $0 \leq \theta \leq 360^\circ$ (or $0 \leq \theta \leq 2\pi$, in radians).
Note that in polar form, two complex numbers are equal if they have the same moduli and their arguments differ by a multiple of 2π.

Conversion of z from polar form to Cartesian form.

Suppose that r (= |z|) and θ are given. Then from trigonometry in triangle Ozx of Figure 3 we find

$$\mathbf{x = r\cos\theta} \qquad (10.1)$$

$$\mathbf{y = r\sin\theta}. \qquad (10.2)$$

Thus $$z = x + iy = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta). \qquad (10.3)$$

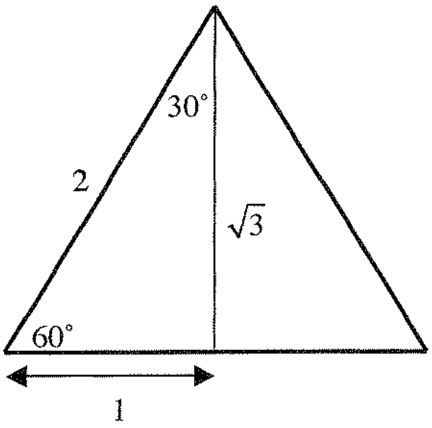
Example :

If $r = 2$ and $\theta = 120^\circ$ then, using a calculator set to degrees, we find

$$x = 2\cos 120^\circ = -1$$

$$y = 2\sin 120^\circ = 1.7321.$$

Note : we can find $\sin 120^\circ = \frac{\sqrt{3}}{2}$ and $\cos 120^\circ = -\frac{1}{2}$ without using a calculator, using a 60° - 30° right angled triangle, plus the basic diagram for defining sine and cosine.



Therefore $$\sin 60^\circ = \frac{\text{opposite}}{\text{hypotenuese}} = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuese}} = \frac{1}{2}$$

But $$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}.$$

Therefore $$x = 2\cos 120^\circ = 2\times\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin 120^\circ = 2\times\frac{\sqrt{3}}{2} = \sqrt{3} = 1.7321.$$

And so $$\mathbf{z = -1+\sqrt{3}\,i}.$$

Conversion of a complex number from Cartesian to polar form.

Given $z = x + iy$, we want to find the modulus r and the argument θ.

1. Calculate r from equation (7.1) is straightforward :

$$\mathbf{r = \sqrt{x^2+y^2}}.$$

2. Find θ by dividing equation (10.2) by equation (10.1) to give

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

That is $$\mathbf{\tan\theta = \frac{y}{x}}. \qquad (10.4)$$

3. Calculate θ.

Use $\boldsymbol{\theta = \arctan\frac{y}{x}}$

Given values of x and y, the idea, from equation (10.4), is to divide these, then to use the "inverse" "tan" functions on your calculator to find θ. Unfortunately, if you do this without thinking, it can give you **wrong answers!!** The problem is that, given any particular number t, there are always two geometrically different angles θ_1 and θ_2 such that both $\tan\theta_1 = t$ and $\tan\theta_2 = t$. These angles differ by π (= 180°) as illustrated in the graph of $y = \tan\theta$ in Figure 4 below.

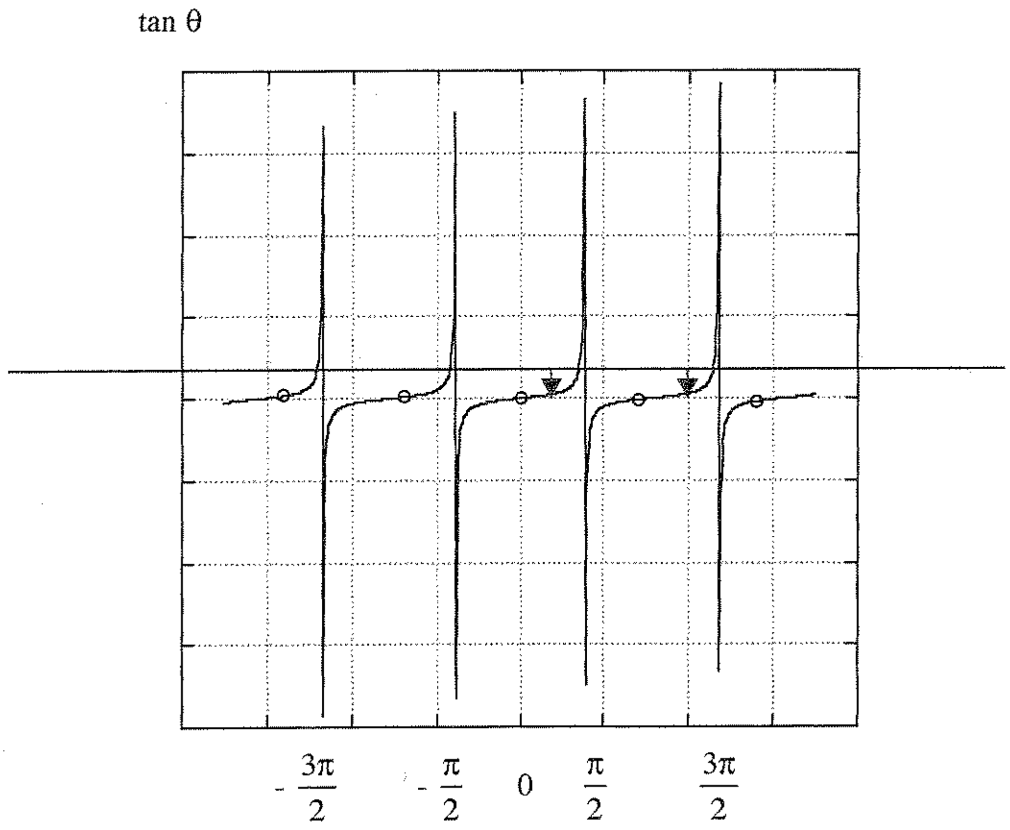


***Figure 4** : Graph of tanθ.*

For example, the two angles ($\theta_1 = 63.4^\circ$ and $\theta_2 = 243.4^\circ$) shown by the arrows both have $\tan\theta = 2$. (This uncertainty is additional to the uncertainty of n(360°) (a whole number of complete circles) discussed above: that uncertainty did not change the geometric angle.)

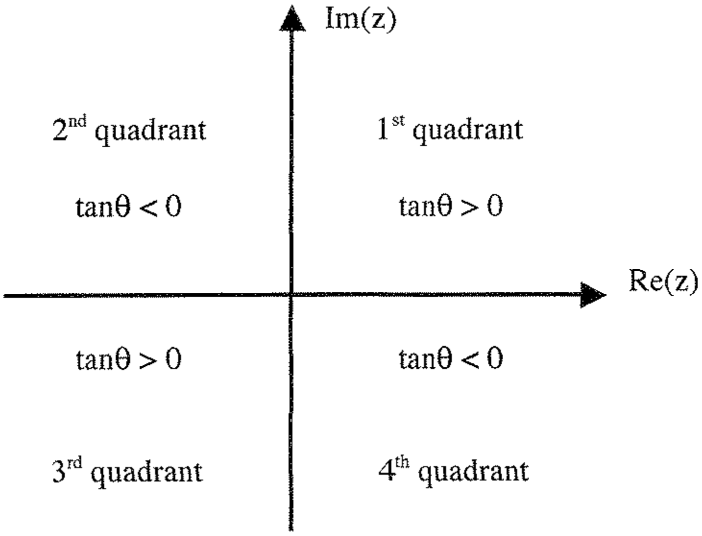
The inverse tangent function on most calculators chooses θ between – 90° and 90° (i.e. between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ radians).

Thus if $\tan\theta > 0$ it will give you a value of θ between 0° and 90° (0 and $\frac{\pi}{2}$ radians)

but if $\tan\theta < 0$ it will give you a value of θ between – 90° and 0° (i.e. between $-\frac{\pi}{2}$ and 0 radians).

However, if $\tan\theta > 0$, θ could either be between 0° and 90° (0 and $\frac{\pi}{2}$ radians) or
between 180° and 270° (π and $\frac{3\pi}{2}$ radians),

whereas if $\tan\theta < 0$, θ could either be between 90° and 180° ($\frac{\pi}{2}$ and π radians) or
between 270° and 360° ($\frac{3\pi}{2}$ and 2π radians).



Thus a calculator may or may not give you the correct value of θ.

The safest way to be sure is to sketch an argand diagram based on the given values of x and y.

Example :

Let $z_1 = 1 + 2i$. Find modulus r_1 and angle θ_1 in degrees.

1. Find the modulus r_1.

$$r_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

2. Calculate $\tan\theta_1$.

$$\tan\theta_1 = \frac{y_1}{x_1} = \frac{2}{1} = 2.$$

3. Use calculator to find θ_1.

Set calculator to degrees.

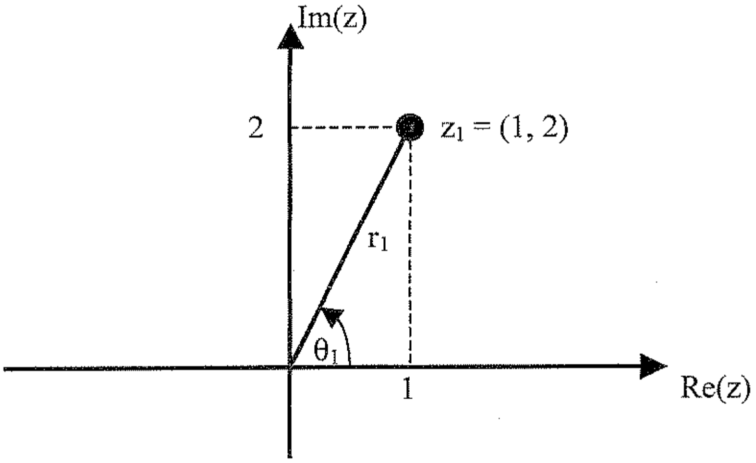
Use your calculator's **tan^{-1}** or **inv tan** button to find

$$\theta_1 = 63.4^\circ .$$

That means $\theta_1 = 63.4^\circ$ or $63.4^\circ + 180^\circ$

that is $\theta_1 = 63.4^\circ$ or 243.4°.

Draw z_1 on the Argand plane.



You can clearly see from the argand diagram that the correct θ is

$$\theta_1 = 63.4^\circ \text{ or } \cancel{243.4^\circ}.$$

Therefore $\mathbf{r_1 = \sqrt{5}}$ **and** $\mathbf{\theta_1 = 63.4^\circ}$.

Example :

Let $z_2 = -1 - 2i$. *Find modulus* r_2 *and angle* θ_2 *in degrees.*

1. Find the modulus r_2.

$$r_2 = \sqrt{x_2^{\,2} + y_2^{\,2}} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{\mathbf{5}} .$$

2. Calculate $\tan\theta_2$.

$$\tan\theta_2 = \frac{y_2}{x_2} = \frac{-2}{-1} = \mathbf{2} .$$

3. Use calculator to find θ_2.

Set calculator to degrees.

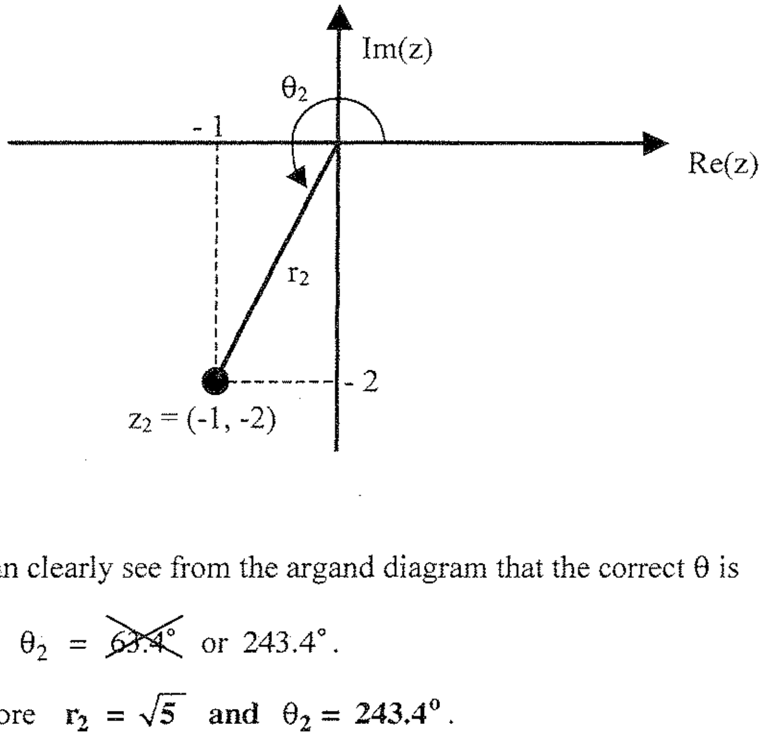
Use your calculator's **tan**$^{-1}$ or **inv tan** button to find

$$\theta_2 = 63.4^\circ .$$

That means $\theta_2 = 63.4^\circ$ or $63.4^\circ + 180^\circ$

that is $\theta_2 = 63.4^\circ$ or 243.4°.

Draw z_2 on the Argand plane.



You can clearly see from the argand diagram that the correct θ is

$$\theta_2 = \cancel{63.4^\circ} \text{ or } 243.4^\circ .$$

Therefore $\mathbf{r_2 = \sqrt{5}}$ **and** $\mathbf{\theta_2 = 243.4^\circ}$.

Example :

Let $z_3 = -1 + 2i$. *Find modulus* r_3 *and angle* θ_3 *in radians.*

1. Find the modulus r_3.

$$r_3 = \sqrt{x_3{}^2 + y_3{}^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{\mathbf{5}}\,.$$

2. Calculate $\tan\theta_3$.

$$\tan\theta_3 = \frac{y_3}{x_3} = \frac{2}{-1} = \mathbf{-2}\,.$$

3. Use calculator to find θ_3.

Set calculator to radians.

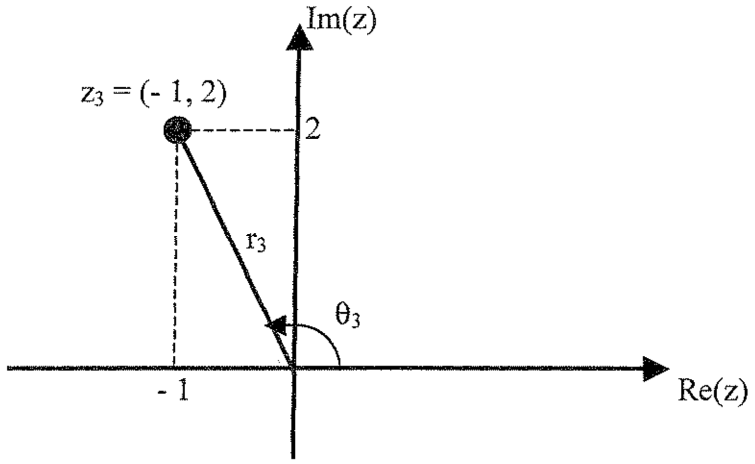
Use your calculator's **tan^{-1}** or **inv tan** button to find

$$\theta_3 = -1.107\,.$$

That means $\theta_3 = -1.107$ or $-1.107 + \pi$

that is $\theta_3 = -1.107$ or 2.034 radians.

Draw z_3 on the Argand plane.



You can clearly see from the argand diagram that the correct θ is

θ_3 = ~~- 1.107~~ or 2.034 radians .

Therefore $\mathbf{r_3 = \sqrt{5}}$ **and** $\boldsymbol{\theta_3} =$ **2.034 radians** .

Example :

Let $z_4 = 1 - 2i$. Find modulus r_4 and angle θ_4 in radians.

1. Find the modulus r_4.

$$r_4 = \sqrt{x_4^2 + y_4^2} = \sqrt{1^2 + (-2)^2} = \sqrt{5}.$$

2. Calculate $\tan\theta_4$.

$$\tan\theta_4 = \frac{y_4}{x_4} = \frac{-2}{1} = -2.$$

3. Use calculator to find θ_4.

Set calculator to radians.

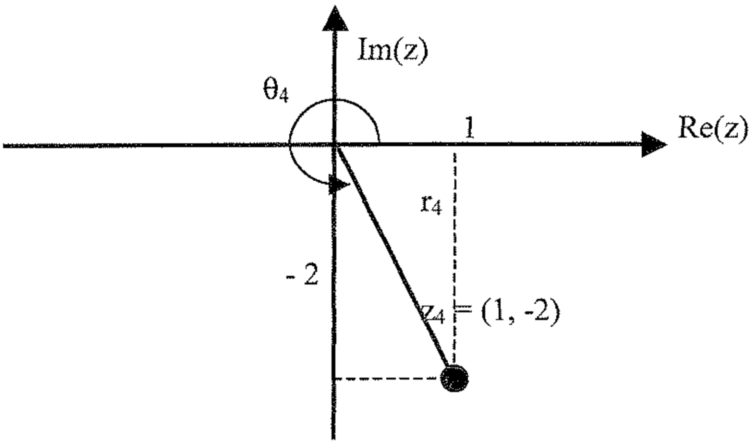
Use your calculator's **tan^{-1}** or **inv tan** button to find

$$\theta_4 = -1.107.$$

That means $\theta_4 = -1.107$ or $-1.107 + \pi$

that is $\theta_4 = -1.107$ or 2.034 radians.

Draw z_4 on the Argand plane.



You can clearly see from the argand diagram that the correct θ is

$$\theta_4 = -1.107 \text{ or } \cancel{2.034} \text{ radians}.$$

Therefore **$r_4 = \sqrt{5}$ and $\theta_4 = -1.107$ radians.**

Some people, however, like to quote angles in the conventional range $0 \le \theta < 2\pi$. This can be achieved by adding 2π radians. This does not change the geometric angle but brings the numerical angle into the conventional range. Thus here $\theta_4 = -1.107 + 2\pi = 5.176$.

Therefore **$r_4 = \sqrt{5}$ and $\theta_4 = 5.176$ radians.**

All this can be summed up as follows:

Calculate $\mathbf{r} = \sqrt{x^2 + y^2}$.

Find $\mathbf{\tan\theta} = \frac{y}{x}$.

Calculate $\boldsymbol{\theta} = \mathbf{arctan}\frac{y}{x}$.

Use inverse tan calculator function to find θ .

Calculate the two possible values of θ.

θ or $\theta + \pi$ radians

Sketch z on the argand diagram to determine which is the correct value of θ.

If you have the correct geometric angle, but is not in the desired range $0 \le \theta < 360^\circ$, add an integer number of 360°.

Important note:

If x = 0, and $y > 0$ then $\theta = 90^\circ = \frac{\pi}{2}$ radians , or

If $y < 0$ then $\theta = 270^\circ = \frac{\pi}{2}$ radians.

11 Multiplying and dividing complex numbers in polar form.

Let $z_1 = x_1 + iy_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = x_2 + iy_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$ (see equation(10.3)). Then

$$\begin{aligned} z_1 z_2 &= |z_1||z_2|(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= |z_1||z_2|(\cos\theta_1\cos\theta_2 + \cos\theta_1 i\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_1 i\sin\theta_2) \\ &= |z_1||z_2|(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \\ &= |z_1||z_2|(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &= |z_1||z_2|(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i[\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2]) \end{aligned}$$

Using the trigonometric sum rules :

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$\sin(\theta_1 + \theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2$$

we can write z_1z_2 as

$$z_1 z_2 = |z_1||z_2|[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \qquad (11.1)$$

Comparing this with equation (10.3) $z = r(\cos\theta + i\sin\theta)$ we see that the complex number $p = z_1z_2$ has modulus $p = |z_1||z_2|$ and argument $\psi = \theta_1 + \theta_2$.
In other words, we can multiply complex numbers by multiplying the moduli and adding the arguments.

Example :

If z_1 has magnitude 3 and argument - π/2 and z_2 has magnitude 2 and argument π/4, find z_1z_2.

Therefore $$z_1 = 3\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$$

$$z_2 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right].$$

Therefore z_1z_2 has magnitude $|z_1||z_2| = 3\times 2 = \mathbf{6}$ and

argument $\theta_1 + \theta_2 = -\frac{\pi}{2} + \frac{\pi}{4} = \mathbf{-\frac{\pi}{4}}.$

Alternatively

$$
\begin{aligned}
z_1 z_2 &= 3\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \\
&= 3\times 2\left[\cos\left(-\frac{\pi}{2}+\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{2}+\frac{\pi}{4}\right)\right] \\
&= 6\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right].
\end{aligned}
$$

Similarly, we can divide complex numbers by dividing the moduli and subtracting the arguments.

That is $$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]. \qquad (11.2)$$

Example :

If z_1 has magnitude 3 and argument $-\pi/2$ and z_2 has magnitude 2 and argument $\pi/4$, find $\frac{z_1}{z_2}$.

Therefore $$z_1 = 3\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$$

$$z_2 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right].$$

Therefore $\frac{z_1}{z_2}$ has magnitude $\frac{|z_1|}{|z_2|} = \frac{3}{2}$ and

argument $\theta_1 - \theta_2 = -\frac{\pi}{2} - \frac{\pi}{4} = -\frac{3\pi}{4}$.

Alternatively,

$$\frac{z_1}{z_2} = \frac{3\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]}{2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]}$$

$$= \frac{3}{2}\left[\cos\left(-\frac{\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$= \frac{3}{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right].$$

12 Exponential function of imaginary argument.

Functions like sin x, cos x and $\exp(x) = e^x$ were defined for real numbers x in 1201SCE Mathematics 1A.

By using Taylor's series, we can develop a meaningful definition of these functions for an arbitrary complex number z.
Taylor's series provides an approximation for a function f(x) in terms of integer powers of the form x^n :

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

For $f(x) = e^x$, $\sin x$ and $\cos x$. evaluation of the derivatives $f, f', f'', \ldots$ at x = 0 gives the Taylor series as :

$$\exp(x) \equiv e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \ldots \tag{12.1}$$

$$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \ldots \tag{12.2}$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots \tag{12.3}$$

$$\cos x = 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \ldots \tag{12.4}$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots \tag{12.5}$$

In addition to being good approximations when a finite number of terms are kept, these particular three power series can be shown to converge exactly to the stated functions, when the number of terms approach infinity.

Multiplying two series together and collecting terms one can use the power series (12.1) to prove the following well-known rule :

$$e^{x_1}e^{x_2} = e^{x_1 + x_2}. \tag{12.6}$$

It then follows by induction that

$$\left(e^x\right)^n = e^{nx}. \tag{12.7}$$

It is important to note that equations (12.6) and (12.7) follow from the power series (12.1) using the laws of algebra alone. Therefore they are valid when we substitute a complex number for z, as explained in the following.

Since (12.1) contains only integer powers of x, we can use it to define e^z for complex values of z.

In particular, for imaginary $z = i\theta$,

$$\begin{aligned}
e^{i\theta} &= 1 + \frac{1}{1!}(i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \ldots \\
&= 1 + i\frac{\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \ldots \\
&= 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \ldots \\
&\qquad\qquad\qquad (\textit{using } i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i) \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right).
\end{aligned}$$

By comparing this with equation (12.5) and (12.3) we then have

$$\mathbf{e^{i\theta} = \cos\theta + i\sin\theta}. \tag{12.8}$$

This is an important result and is known as **Euler's Formula**.

In a similar fashion it can be shown that

$$\exp(-i\theta) \equiv e^{-i\theta} = \cos\theta - i\sin\theta. \tag{12.9}$$

Equation (12.8) can alternatively be expressed as

$$\mathrm{Re}\left(e^{i\theta}\right) = \cos\theta, \qquad \mathrm{Im}\left(e^{i\theta}\right) = \sin\theta.$$

From equation (10.3) and (12.8) we can now write a general complex number $z = x + iy$ as

$$\begin{aligned}
z = x + iy = r\cos\theta + ir\sin\theta &= r(\cos\theta + i\sin\theta) \\
&= re^{i\theta} = |z|e^{i\theta}.
\end{aligned}$$

Therefore the two main forms of expressing a complex number is

$$\mathbf{z = \underbrace{x + iy}_{\text{cartesian form}} = \underbrace{re^{i\theta}}_{\text{polar form}}}.$$

Equations (12.8) and (12.9) gives

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta .$$

Adding these equations gives

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta \\ &= 2\cos\theta . \end{aligned}$$

Therefore dividing both sides by 2 gives

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right). \qquad (12.10)$$

Subtracting (12.9) from (12.9) gives

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= \cos\theta + i\sin\theta - \cos\theta + i\sin\theta \\ &= 2i\sin\theta . \end{aligned}$$

Therefore dividing both sides by 2i gives

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right). \qquad (12.11)$$

The combination $\cos\theta + i\sin\theta$ is often denoted "cis θ" so that

$$\text{cis}\theta = \cos\theta + i\sin\theta = \exp(i\theta) = e^{i\theta} .$$

There is really no need for this additional notation, and it will not be used further here. It is much more helpful to use the notation $\exp(i\theta)$ or $e^{i\theta}$ because this reminds us that for all the useful properties of the exponential function can be employed, as we will see.

The exponential of a complex number, $z = x + iy$ is, from equations (12.6) and (12.8),

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y). \qquad (12.12)$$

13 Multiplying complex numbers in polar form : II.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i\theta_1} e^{i\theta_2}.$$

Now equation (12.6) shows that $e^{\theta_1} e^{\theta_2} = e^{\theta_1 + \theta_2}$, and the exponent can be factored to give $e^{i(\theta_1 + \theta_2)}$. Thus the above product can be written as

$$z_1 z_2 = r_1 r_2 e^{i\theta_1 + i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

In other words, to multiply two complex numbers we can

1. multiply the moduli, and

2. add the arguments.

This just gives us a different version of equation (11.1) which was derived from the addition formulae for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$.

Alternatively, we can use the present method to obtain those addition formulae without having to remember them:

$$\begin{aligned}
\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) &= e^{i(\theta_1 + \theta_2)} = e^{i\theta_1} e^{i\theta_2} \\
&= \left(\cos\theta_1 + i\sin\theta_1\right)\left(\cos\theta_2 + i\sin\theta_2\right) \\
&= \cos\theta_1 \cos\theta_2 + \cos\theta_1 i\sin\theta_2 + i\sin\theta_1 \cos\theta_2 + i\sin\theta_1 i\sin\theta_2 \\
&= \cos\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2 + i^2 \sin\theta_1 \sin\theta_2 \\
&= \cos\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \\
&= \left(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2\right) + i\left(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2\right)
\end{aligned}$$

where we used multiplication of complex numbers in Cartesian (x, y) form.

Equating the real parts of the LHS and RHS gives

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

and equating the imaginary parts gives

$$\sin(\theta_1 + \theta_2) = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2.$$

14 De Moivre's Theorem: Trigonometric identities.

Equation (12.7) gives

$$\left(e^{i\theta}\right)^n = e^{in\theta}.$$

From equation (12.8) we have $e^{i\theta} = \cos\theta + i\sin\theta$

and $e^{in\theta} = \cos n\theta + i\sin n\theta$.

Substituting these into the above equation gives

$$\boldsymbol{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}.$$

This result is known as **De Moivre's Theorem**.

Although it is expressed in complex terms, it can be used to obtain trigonometric results for real numbers.

Example :

Find an expression for cos3θ involving powers of cosθ.

Put n = 3 in De Moivre's Theorem, above. That is

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta .$$

Taking the real part of both sides we obtain

$$\mathrm{Re}(\cos 3\theta + i\sin 3\theta) = \mathrm{Re}\left[(\cos\theta + i\sin\theta)^3\right]$$

$$\therefore \quad \cos 3\theta = \mathrm{Re}\left[\cos^3\theta + 3(\cos^2\theta)(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3\right]$$

(multiplying out the brackets)

$$= \mathrm{Re}\left[\cos^3\theta + i3(\cos^2\theta)(\sin\theta) + i^2 3\cos\theta(\sin\theta)^2 + i^3\sin^3\theta\right]$$

$$= \mathrm{Re}\left[\cos^3\theta + i3(\cos^2\theta)(\sin\theta) - 3\cos\theta(\sin\theta)^2 - i\sin^3\theta\right]$$

(using $i^2 = -1$, $i^3 = -i$)

$$= \mathrm{Re}\left[\left(\cos^3\theta - 3\cos\theta(\sin\theta)^2\right) + i\left(3(\cos^2\theta)(\sin\theta) - \sin^3\theta\right)\right]$$

(collecting real and imaginary terms)

$$= \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= \cos^3\theta - 3\cos\theta\left(1 - \cos^2\theta\right) = \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$

(using $\sin^2\theta = 1 - \cos^2\theta$)

$$= \mathbf{4\cos^3\theta - 3\cos\theta} \ .$$

In general, the connection between imaginary exponentials and real trigonometric functions is a powerful tool for deriving trigonometric identities.

Example :

Find an expression for $\sin^3\theta$ *in terms of the trig functions of multiple angles.*

Using equation (12.11)

$$\begin{aligned}\sin^3\theta &= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^3\\ &= \frac{1}{(2i)^3}\left[\left(e^{i\theta}\right)^3 - 3\left(e^{i\theta}\right)^2\left(e^{-i\theta}\right) + 3\left(e^{i\theta}\right)\left(e^{-i\theta}\right)^2 - \left(e^{-i\theta}\right)^3\right]\\ &= \frac{1}{-8i}\left[e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}\right]\\ &= -\frac{1}{4}\left[\frac{e^{3i\theta} - e^{-3i\theta}}{2i} - \frac{3\left(e^{i\theta} - e^{-i\theta}\right)}{2i}\right]\\ &= \boldsymbol{-\frac{1}{4}(\sin 3\theta - 3\sin\theta)}\ .\end{aligned}$$

Example :

Use the above to find the indefinite integral $\int \sin^3 x\, dx$.

From above,

$$\begin{aligned}\int \sin^3 x\ dx &= \int -\frac{1}{4}(\sin 3x - 3\sin x)\ dx\\ &= -\frac{1}{4}\int \sin 3x\ dx + \frac{3}{4}\int \sin x\ dx\\ &= -\frac{1}{4}\left(-\frac{1}{3}\cos 3x\right) + \frac{3}{4}(-\cos x) + C\\ &= \boldsymbol{\frac{1}{12}\cos 3x - \frac{3}{4}\cos x + C}\ .\end{aligned}$$

Another way to do the integral $\int \sin^3 x \, dx$ is by substitution using

$u = \cos x$, so that $\dfrac{du}{dx} = -\sin x$ and so $dx = \dfrac{du}{-\sin x}$.

$$\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \sin x \, dx$$

$$= \int (1 - u^2) \sin x \frac{du}{-\sin x} \qquad \textit{(rewriting in terms of u)}$$

$$= - \int (1 - u^2) \, du \qquad \textit{(simplifying)}$$

$$= - \left(u - \frac{1}{3} u^3 \right) + C$$

$$= -u + \frac{1}{3} u^3 + C \qquad \textit{(simplifying)}$$

$$= \mathbf{-\cos x + \frac{1}{3} \cos^3 x + C} \qquad \textit{(substituting u = cos x)}$$

But from the first example $\quad \cos 3x = 4\cos^3 x - 3\cos x$

which gives $\quad \cos^3 x = \dfrac{1}{4}(\cos 3x + 3\cos x)$.

Therefore substituting this above gives

$$\int \sin^3 x \, dx = -\cos x + \frac{1}{3} \times \frac{1}{4} (\cos 3x + 3\cos x) + C$$

$$= -\cos x + \frac{1}{12} \cos 3x + \frac{1}{4} \cos x + C$$

$$= \mathbf{\frac{1}{12} \cos 3x - \frac{3}{4} \cos x + C}.$$

which agrees with the above.

15 nth roots of a complex number.

Imaginary numbers were originally invented to solve equations such as

$$w^2 = -6,$$

but in fact complex numbers permit solutions of the more general equation

$$w^n = z \tag{15.1}$$

where z is any complex number and n is a positive integer. One solution is obtained immediately from the second exponential law equation (12.7), $(e^x)^n = e^{nx}$,

by raising both sides of equation (15.1) to the power 1/n

$$w = w^{n\frac{1}{n}} = (w^n)^{\frac{1}{n}} = z^{\frac{1}{n}}.$$

This operation can be evaluated using the polar form of z :

$$w = \left(|z|e^{i\theta}\right)^{1/n} = |z|^{1/n}e^{i\theta/n} = \sqrt[n]{|z|}\,e^{i\theta/n}.$$

That is, w has a modulus to the nth root of the modulus of z, and an argument 1/n times the argument of z. For example, with n = 3, a cube root ("3th root") of

$$z = -8 = 8e^{i\pi} \text{ is}$$

$$w = \left(8e^{i\pi}\right)^{1/3} = 8^{1/3}(e^{i\pi})^{1/3} = 2e^{i\pi/3}.$$

Although this procedure gives us one nth root, in general there are n different nth roots of a given nonzero complex number. (Eg. There are 2 square roots (2th roots) of – 1 : they are + i and – i). There are 3 cube roots of – 8.

To obtain all nth roots of z, we make use of the ambiguity of the argument discussed in Section 7 : we can add an arbitrary integer number of whole circles to the argument of z, without altering the number z .

Thus if $z = |z|e^{i\theta}$

then we can also write $z = |z|e^{i(\theta + 2\pi m)}$

where m is an arbitrary integer $(m = 0, \pm 1, \pm 2, \pm 3, \ldots)$,

Thus if z is a non-zero complex number and n is a positive integer, the equation

$$w^n = z$$

has n distinct solutions for w, known as the nth roots of z.

These can be found by writing z in the form

$$z = |z| e^{i(\theta + 2\pi m)}$$

and raising to the $\frac{1}{n}$th power, giving

$$w = |z|^{1/n} e^{i\frac{\theta + 2m\pi}{n}} \qquad (15.2)$$

Using n consecutive values of m will give n distinct roots of w (eg. for n = 5, the values m = - 2, - 1, 0, 1, 2 will suffice).

Example :

Find all the 6^{th} roots of 1, that is $\sqrt[6]{1}$. Another way to write this is, if $w^6 = 1$, find all w.

1. Express 1 in polar form.

$$\text{Here } z = 1 = 1e^{i0} = 1e^{i(0 + 2m\pi)} .$$

2. Find an expression for w in polar form.

$$\text{If } w^6 = 1$$

$$\text{then } w^6 = 1e^{i(0 + 2m\pi)} \text{ and therefore}$$

$$w = \left(1e^{i(0 + 2m\pi)}\right)^{1/6} = 1^{1/6}e^{i(0 + 2m\pi)/6} = 1e^{im\pi/3}.$$

3. Calculate the 6 roots.

Choose 6 consecutive values of m and substitute into the expression for w above.

Here we will choose $m = -2, -1, 0, 1, 2, 3$ but we could have chosen values of m to be $m = 0, 1, 2, 3, 4, 5$ or $m = -5, -4, -3, -2, -1, 0$.

Choose $m = -2$: This gives one sixth root $w_{-2} = e^{i\pi(-2)/3} = e^{-(2/3)\pi i}$

Choose $m = -1$: This gives one sixth root $w_{-1} = e^{i\pi(-1)/3} = e^{-(1/3)\pi i}$

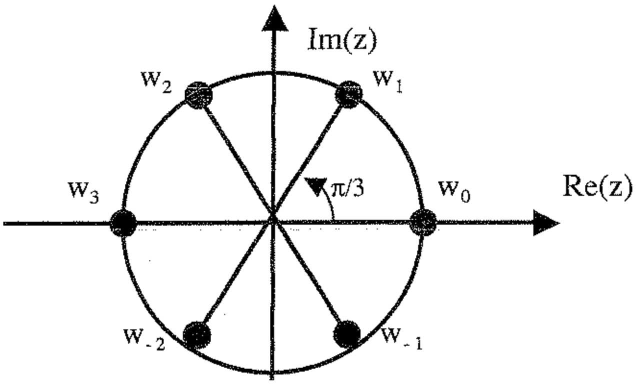
Choose $m = 0$: This gives one sixth root $w_0 = e^{i\pi(0)/3} = e^{(0)\pi i} = 1$

Choose $m = +1$: This gives one sixth root $w_{+1} = e^{i\pi(1)/3} = e^{(1/3)\pi i}$

Choose $m = +2$: This gives one sixth root $w_{+2} = e^{i\pi(2)/3} = e^{(2/3)\pi i}$

Choose $m = +3$: This gives one sixth root $w_{+3} = e^{i\pi(3)/3} = e^{(3/3)\pi i} = e^{\pi i} = -1$.

Note that these 6 roots are equally spaced around the unit circle, each at angle $\pi / 3 = 60° = 360° / 6$ to the previous one.



If we had continued on with m = 4, we would have found a root as follows:

Choose $m = +4$: This gives one sixth root $w_{+4} = e^{i\pi(4)/3} = e^{(4/3)\pi i}$.

This root w_{+4} with angle $\frac{4\pi}{3} = 240°$ is identical with w_{-2} at $-\frac{2\pi}{3} = -120°$, as can be seen by adding a complete circle 2π radians or $360°$.

The six complex roots are unique, but the quoted numerical values of the angles are not unique. We obtained the 6 angles closest to zero by choosing $m = 0, \pm 1, \pm 2, 3$. If we wanted to conform to the common convention that the angles are between 0 and 2π, we could have chosen $m = 0, 1, 2, 3, 4, 5$.

Example :

Find all the fourth roots of $z = -1 + i$.

Another way to write this is, if $w^4 = -1+i$, *find all values of* w.

1. Express $-1 + i$ in polar form.

Here $z = -1 + i = Re^{i(\theta + 2\pi m)}$.

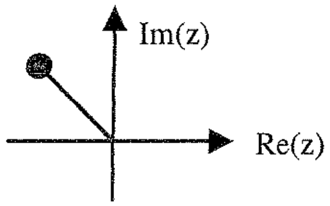
Need to find R and θ.

$$R = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \text{ and } \tan\theta = \frac{1}{-1} = -1.$$

$$\theta = \tan^{-1}(-1)$$

$$\therefore \theta = -\frac{\pi}{4} \text{ or } -\frac{\pi}{4} + \pi = -\frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

Draw argand diagram to find θ.



$$\therefore \theta = \xcancel{-\frac{\pi}{4}} \text{ or } \frac{3\pi}{4}$$

$$\therefore R = \sqrt{2} \text{ and } \theta = \frac{3\pi}{4}$$

$$\therefore w^4 = \sqrt{2}e^{i(3\pi/4 + 2\pi m)}$$

2. Find an expression for w in polar form.

If $w^4 = -1 + i$

then $w^4 = \sqrt{2}e^{i(3\pi/4 + 2\pi m)}$ and therefore

$$w = \left(\sqrt{2}e^{i(3\pi/4 + 2\pi m)}\right)^{1/4} = \left(\sqrt{2}\right)^{1/4} e^{i(3\pi/4 + 2\pi m)/4}$$

$$= \left(2^{1/2}\right)^{1/4} e^{i(3\pi/4 + 2\pi m)/4} = 2^{1/8}e^{i(3\pi/4 + 2\pi m)/4}.$$

3. Calculate the 4 roots.

Choose 4 consecutive values of m and substitute into the expression for w above.

This time we will choose $m = 0, 1, 2, 3$ to give angles in the range $0 \le \theta < 2\pi$.

Choose $m = 0$: This gives one sixth root

$$w_0 = 2^{1/8}e^{i(3\pi/4 + 2\pi 0)/4} = 2^{1/8}e^{i(3\pi/4 + 0)/4} = 2^{1/8}e^{3\pi i/16}$$

Choose $m = 1$: This gives one sixth root

$$w_1 = 2^{1/8}e^{i(3\pi/4 + 2\pi 1)/4} = 2^{1/8}e^{i(3\pi/4 + 2\pi)/4} = 2^{1/8}e^{11\pi i/16}$$

Choose $m = 2$: This gives one sixth root

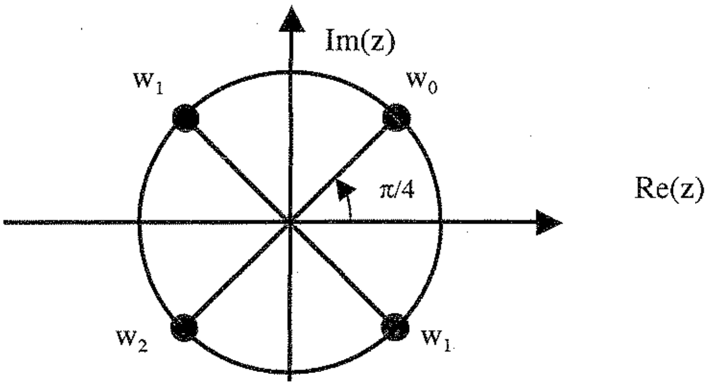
$$w_2 = 2^{1/8}e^{i(3\pi/4 + 2\pi 2)/4} = 2^{1/8}e^{i(3\pi/4 + 4\pi)/4} = 2^{1/8}e^{19\pi i/16}$$

Choose $m = 3$: This gives one sixth root

$$w_3 = 2^{1/8}e^{i(3\pi/4 + 2\pi 3)/4} = 2^{1/8}e^{i(3\pi/4 + 6\pi)/4} = 2^{1/8}e^{27\pi i/16}.$$

Note that these 4 roots all have magnitude $2^{1/8} = 1.091$.

Consecutive roots are equally spaced around the unit circle, each at angle $2\pi / 4 = 90^\circ = \pi / 2$ to the previous one.



Example:

Express the root w_3 from the previous example in Cartesian form.

From equation (10.3) $\quad z = x + iy = r(\cos\theta + i\sin\theta)$.

Therefore $\quad w_3 = |w_3|(\cos\theta + i\sin\theta)$.

But we know that $|w_3| = 2^{1/8}$ and $\theta_3 = 27\pi/16$

Therefore

$$\begin{aligned} w_3 &= 2^{1/8}[\cos(27\pi/16) + i\sin(27\pi/16)] \\ &= 1.0905[0.55557 + i(-0.83147)] \\ &= \mathbf{0.60585 - 0.90672i}. \end{aligned}$$