1- $f(x) = x^3 - 3x^2 + 27$

a- The only Solution of $f(x) \equiv 0 \pmod 3$ is $x \equiv 0 \pmod 3$. Give a very short proof. (not using Hensl's Lamma) that

$f(x) \equiv 0 \pmod{3^k}$ has no solution fo $k \geq 4$.

b- Use Hensel's Lamma and the Chinese Remaincler Theorem to find all solution of $f(x) \equiv 0 \pmod{1125}$ (Show your methods of collculation)

2-a- If $w(n)$ = number of distinct prime factor of n,

$$\sum_{k/n} \mu(k)\, d(k) = (-1)^{w(n)}$$

b- If f and g are both multiplicative prove that

$$h(n) = \sum_{d/n} f(d)\, g\left(\frac{n}{d}\right)$$ is multiplicative ~~and $M_p = 2^p - 1$~~

~~is nonprime messene number, prove that M_p is a pseudoprime to base 2 that is,~~

$\cancel{2^{M_p} \equiv 2 \pmod{M_p}}$

3- If p is a prime and $M_p = 2^p - 1$ is nonprime messene number, prove that M_p is a pseudoprime to base 2, that is $2^{M_p} \equiv 2 \pmod{M_p}$.

4- $$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots\right) = \frac{\pi^2}{6}$$

a- Show that $\zeta(2)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$

$$\left(\text{Equivalently } \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} = 1\right)$$

b- Use the result of problem 6 in section 6-4 to prove that the number of sequer free integers k $1 \leq k \leq n$,

$$S(n) = \frac{6n}{\pi^2} + O(\sqrt{n})$$

(note: $f(n) = O(g(n))$ if $|f(n)| \leq c\, g(n)$ for some constant c and all $n \geq 1$).