Thus Theorem 6–8 now follows from Definition 6–3. ■

Noting that the functions $f_1(n) = 1$ and $f_2(n) = n$ are multiplicative, we deduce from Theorems 6–7 and 6–8 that $\phi(n)$, $d(n)$, and $\sigma(n)$ are also multiplicative (an alternative proof of Theorem 6–4).

EXERCISES (* means difficult)

1. Prove that if $\sigma_k(n) = \sum_{d \mid n} d^k$, then $\sigma_k(n)$ is multiplicative.

2. Prove that if $f(n)$ is multiplicative, then
$$\sum_{d \mid n} \mu(d) f(d) = \prod_{p \mid n} \{1 - f(p)\}.$$

3. Use Exercise 2 to give a new proof of the second part of Theorem 6–2; namely, that
$$\sum_{d \mid n} \mu(d) \frac{n}{d} = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

4. Let $S(n)$ denote the number of square-free integers (see Exercise 13 of Section 6–2) not exceeding n. Prove that
$$S(n) = \sum_{j=1}^{n} |\mu(j)|.$$

*5. With the definition of $S(n)$ given in Exercise 4, prove that
$$S(n) = \sum_{j=1}^{n} \sum_{d^2 \mid j} \mu(d). \text{ [Hint: Use Theorem 6–5.]}$$

*6. Use Exercise 5 to prove that
$$S(n) = \sum_{d=1}^{n} \mu(d) \left[\frac{n}{d^2}\right],$$
where $\left[\frac{n}{d^2}\right]$ denotes the largest integer that does not exceed $\frac{n}{d^2}$.