Set 1

Question 1

(a) Consider the equation $f(x) = 0$. Suppose a root $x = \alpha$ lies in the interval $[a_0, b_0]$ and it is desired to find the root to n decimal places of accuracy. Using the bisection method, show the following.

(i) $k \geq \{n + \log_{10} |b_0 - a_0|\} / \log_{10} 2$.

(Assumming $f(\frac{1}{2}(a_0 + b_0) \neq 0)$ where k is the smallest number of bisections.)

(ii) At most $k + 1$ iterations are needed.

(iii) At most $k + 3$ function evaluations are needed.

(b) Given

$$\exp(x) + x - 2 = 0 \qquad (1)$$

Confirm that there is a root in $[0, 0.5]$, and use the bisection method to show that $\alpha = 0.4$. (correct to 1 dp of accuracy)

[NB. Use the following table format for your root location.]

r	a_r	b_r	c_r	$f(a_r)$	$f(b_r)$	$f(c_r)$	$b_r - a_r$
0 1 : k							

(c) A simple fixed point iterative formula of eqn (1) is given by

$$x_{r+1} = f(x_r) = 2 - \exp(x_r). \qquad (2)$$

(i) Show that the iterative scheme diverges by computing the first five iterations, starting with $x = 0.4$.

(ii) Show that the function $f(x)$ has no contraction mapping on $[0, 0.5]$.

(iii) Show that using the Add-nx method an equivalent formula to eqn (2) can be written as

$$x_{r+1} = g(x_r) = 0.6x_r - 0.4\exp(x_r) + 0.8 \qquad (3)$$

where $g(x)$ has a contraction mapping on [0, 0.5]. Hence compute the root of eqn (1) correct to 5 dp of accuracy. (Starting with $x_0 = 0.4$, all subsequent values x_1, x_2, x_3, etc must be shown.)

Question 2

(a) A root of the equation

$$f(x, c) = \exp(-cx^2) + cx - 3.011 = 0$$

is $\alpha = 1.5$ when $c = 2.0$.

Determine the absolute conditioning and relative conditioning of the problem of finding this root if c is subject to small changes about the value 2.0, by finding

(i) the absolute condition number κ_a.

(ii) the relative condition number κ_r.

(iii) Estimate the new root when c changes to 2.01, by using the condition number.

(b) The Newton-Raphson formula for finding a root of the equation $f(x) = 0$ is

$$x_{r+1} = x_r - \{f(x_r)/f'(x_r)\}$$

(i) Considering the formula as a simple iterative scheme, use the contraction mapping principle to show that the iteration converges for any interval $[a_0, b_0]$ where the root lies in.

(ii) Use the formula to compute the new root of the equation

$$\exp(-2.01x^2) + 2.01x - 3.011 = 0$$

in part(a) above, using only 1 iteration, starting with $x_0 = 1.5$. Comment and contrast the two results.

Question 3

(a) The rate of fuel consumption by an experimental aircraft engine is given by a known unimodal function

$$f(x) = (2-5x)\exp(3-4x) + 5$$

where x denotes the position of a knob on a calibrated control valve. From trial and error minimal consumption rate is found when the knob position lies in the interval [0.5, 1].

(i) Perform three iterations, using the golden section search algorithm, to locate a minimizer for $f(x)$. (You need not explain the workings of the algorithm, but you must record your results in table form).

(ii) For each iteration, state and give reasons the interval in which the minimizer lies in. At the end of the iterations, explain how the new interval for the minimzer is [0.57, 0.69].

(b) A mountainous landscape can be described by a 2-dimensional variable function

$$f(x, y) = 16x_1^2 + 26x_2^2 - 40x_1x_2 - 20x_2 + 107.$$

A mathematics student on a climbing expedition ponders how many iterations it would take for the steepest descent algorithm to find the bottom.

(i) Assuming his current coordinate is $(1, 1)^t$, perform 1 iteration of the steepest descent algorithm to locate a minimizer for $f(x, y)$. State the new coordinate of the student, and the function value, assuming he moves to the new position at the end of the iteration. Briefly describe how the minimum of $f(x, y)$ can then be found.

(ii) Find the minimizer and minimal value for $f(x, y)$ using a non-iterative method.

Question 4

A system of nonlinear equations is given by

$$f_1(\mathbf{x}) = x_1 + x_2^2 - 0.1 = 0$$
$$f_2(\mathbf{x}) = x_1^2 + x_2 - 0.2 = 0.$$

(a) Sketch graphs for the above functions on the x_1-x_2 plane and show that there are two roots to the equations. Verify numerically that the roots $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are approximately $(0.1, 0.2)^t$ and $(-1.1, -1.1)^t$, respectively.

(b) An iterative scheme $\mathbf{x}^{(r+1)} = \mathbf{g}(\mathbf{x}^{(r)})$ is to be adopted for the above equations. Show that $\mathbf{g}(\mathbf{x})$ can be expressed in any of the following forms

S1	S2	S3
$g_1(\mathbf{x}) = 0.1 - x_2^2$	$g_1(\mathbf{x}) = \sqrt{0.2 - x_2}$	$g_1(\mathbf{x}) = -\sqrt{0.2 - x_2}$
$g_2(\mathbf{x}) = 0.2 - x_1^2$	$g_2(\mathbf{x}) = \sqrt{0.1 - x_1}$	$g_2(\mathbf{x}) = -\sqrt{0.1 - x_1}$

(c) Regions R_1 and R_2 are defined by

$$R_1 = \{\mathbf{x} \mid 0 \le x_1 \le 0.2,\ 0.1 \le x_2 \le 0.3\}$$
$$R_2 = \{\mathbf{x} \mid -2 \le x_1 \le -0.5,\ -1.5 \le x_2 \le -0.1\}.$$

Show by your workings using the result of contraction mapping that

(i) starting in R_1 the root $\boldsymbol{\alpha}_1$ can be found iteratively with $\mathbf{g}(\mathbf{x})$ given in scheme S_1.

(ii) starting in R_2 the root $\boldsymbol{\alpha}_2$ cannot be guaranteed to be found iteratively with $\mathbf{g}(\mathbf{x})$ given in scheme S_1.

(d) Determine which region R is to be used in conjunction with which scheme S to find the root $\boldsymbol{\alpha}_2$. (NB. All workings must be clearly shown)

Question 5

An optimization problem with an equality constraint is modelled as

$$\begin{aligned} &\text{minimize} \quad f(\mathbf{x}) = 3x_1^2 + 2x_2^2 - 12x_1 - 12x_2 + 30 \\ &\text{subject to} \quad c(\mathbf{x}) = 3x_1 + 2x_2 - 1 = 0. \end{aligned}$$

(a) The problem is to be solved using the simple penalty-function method.

(i) Write down the quadratic penalty function $\phi(\mathbf{x},\sigma)$ for the model, hence determine a stationary point $\boldsymbol{\alpha}(\sigma)$ for $\phi(\mathbf{x},\sigma)$ in terms of the parameter σ.

(ii) Derive the Hessian matrix for $\phi(\mathbf{x},\sigma)$, and show that $\boldsymbol{\alpha}(\sigma)$ is a minimizer for certain values of σ.

(iii) Compute $\boldsymbol{\alpha}(\sigma)$ and $\phi(\mathbf{x},\sigma)$ for $\sigma = 10^4$, hence deduce a minimizer and minimal value for $f(\mathbf{x})$.

(b) Form the Lagrangian function $L(\mathbf{x},\mu)$ for the model and use it to determine a minimizer for $f(\mathbf{x})$.

Set 2

Question 1

The insulin level in a particular diabetes patient's bloodstream over time in hours following an injection is given by the following model:

$$I(t) = \frac{0.4t + 0.4}{e^{0.5t} + 1}$$

The patient is monitored to keep the insulin level above 0.2.

(a) Show that the range of times when the insulin level falls below 0.2 is found by solving the equation:

$$e^{0.5t} - 2t - 1 = 0.$$

(b) By drawing a suitable sketch or otherwise, show that there are just two non-negative roots of this equation and that these lie in the time interval [0, 6]. State the values of these roots to one decimal place.

(c) Show that a fixed point of the iterative scheme:

$$t_{r+1} = g(t_r) = 2\ln(2t_r + 1)$$

is also a root of the equation in part (a).

(d) Prove that the iterative scheme in (c) satisfies the condition for a contraction mapping on an interval surrounding the larger positive root.

Question 2

The relative position of two electrically charged particles within a Cartesian plan is described by the following system of nonlinear equations:

$$f_1(\mathbf{x}) = 4x_1 - x_1x_2 - 1 \;= 0$$
$$f_2(\mathbf{x}) = x_1^2 + x_2^2 - 9 \quad = 0$$

(a) By drawing a suitable sketch or otherwise, show that there is only one positive root of this system in the intervals [0.5, 1] and [2.5, 3] for x_1 and x_2 respectively.

(b) Write down the Jacobi iterative scheme for this system of equations in the order given, and show that it is a contraction mapping on the region:

$$\mathrm{R} = \{(x_1, x_2): \; 0.6 \le x_1 \le 1.2, \; 2.5 \le x_2 \le 3\}.$$

(c) Carry out the Jacobi iterative scheme, found in (b), for five iterations, starting from $\mathbf{x}^{(0)} = [0.8, 2.5]^T$, and using 5 decimal places accuracy.

Question 3

Consider the following constrained optimization model:

$$\begin{aligned} &\text{minimize} && f(\mathbf{x}) = 2x_2^2 + 4x_1^2 - 6x_1x_2 - 2x_2 + 26x_1 \\ &\text{subject to} && c(\mathbf{x}) = 4x_1 + x_2 - 7 = 0. \end{aligned}$$

(a) Write down the quadratic penalty function of this model. Calculate the value of the approximate local minima for $\sigma = 100$.

(b) Write down the Lagrangian function of the model. Using the necessary conditions for a local minima, deduce the constrained stationary point $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \lambda]^T$. Compare the stationary point $\boldsymbol{\alpha}$ to the local minima found in part (a).

(c) Estimate the change in the minimum value of f if the constraint constant term is changed from 7 to 6.95.

Question 4

Consider the nonlinear unconstrained objective function model:

$$f(\mathbf{x}) = 5x_1^2 + 2x_2^3 + 2x_1x_2 - 12x_1 - 8x_2 .$$

(a) Use two iterative steps of any numerical method to find an approximation to a local minimizer of f, starting from the origin, $\mathbf{x}^{(0)} = [0,0]^{\mathrm{T}}$.

(b) Use an analytical method to find the local minimizers of the function f.

(c) Name two advantages of the BFGS method over the rank one method and one disadvantage of the Newton-Raphson method.

Question 5

Consider the following constrained minimization model:

$$\text{minimize } f(\mathbf{x}) = 2x_1^2 + x_2^2 - 4x_1 - 2x_2 + 3$$
$$\text{subject to } c(\mathbf{x}) = 2x_1 + x_2 - 9 = 0 .$$

(a) Write down an expression for the augmented Lagrangian function $\phi(x,\mu,\sigma)$. Determine an expression for the gradient $\nabla\phi(x,\mu,\sigma)$ and the Hessian matrix $G(\sigma)$. Prove that $G(\sigma)$ is positive definite.

(b) Write down the Lagrangian function for this model and use it to determine the first order necessary conditions. Deduce the stationary point $\boldsymbol{\alpha}$ and the Lagrange multiplier λ.

(c) Show that $\boldsymbol{\alpha}$ is a constrained local minimizer for the model.

Set 3

Question 1

The concentration levels of a PCB pollutant discharged accidentally into Singapore River over time in days is given by the following expression:

$$p(t) = (4t + 10) \times e^{-\frac{1}{4}t}.$$

The Ministry of Environment has installed special sensors at critical areas to monitor the concentration of the pollutant in the water. A concentration level of 2 units or less of PCB per unit volume is considered ecologically safe.

(a) By drawing a suitable sketch, or otherwise, show that the PCB pollutant concentration level in the water would not reach a safe level for at least two weeks.

(b) Given the equation:

$$e^{0.25t} - 2t - 5 = 0.$$

Show that a fixed point of the iterative scheme:

$$t_{r+1} = g(t_r) = 4\ln(2t_r + 5)$$

is also a root of the above equation.

(c) Prove the iterative scheme in (b) satisfies the conditions of a contraction mapping on an interval surrounding the positive root of the equation in part (b).

Question 2

Consider the following equation:

$$f(x) = x^3 - 5x + c.$$

(a) Find the intervals in which all the roots of the equation $f(x) = 0$ are absolutely ill-conditioned with respect to small changes in the constant c.

(b) When the constant term is c = 4.3, the root of the equation $f(x) = 0$ is $\alpha = 1.26$ to 2 decimal places accuracy. Determine whether or not the problem of finding this root is

(i) absolutely ill-conditioned.

(ii) relatively ill-conditioned.

(c) The bisection method failed when it was carried out using a computer optimization software to find the root of $f(x)$ when the constant term c = 4.303, on an interval [1.25, 1.31]. Explain the reason.

Question 3

The position of two particles within a Cartesian plan is described by the following system of nonlinear equations:

$$f_1(\mathbf{x}) = x_1^2 + x_2^2 - 4 = 0$$
$$f_2(\mathbf{x}) = \ln x_1 + x_2 - 1 = 0.$$

(a) By drawing a suitable sketch or otherwise, show that the system has only two positive roots in the interval [0, $e = 2.718$] for x_1 and x_2 respectively.

(b) Write down a Jacobi iterative scheme and specify a suitable rectangular region R, so that the iterative scheme is guaranteed to converge to the smallest root for the above system of nonlinear equations.

(c) Write down a Jacobi iterative scheme and specify a suitable rectangular region R, so that the iterative scheme is guaranteed to converge to the largest root for the above system of nonlinear equations.

Question 4

The production cost of a chemical product is related to the quantity x produced as given by the following model:

$$f(x) = x^4 + e^{-2x}$$

(a) The quantity produced is strictly kept in the range of 0.2 to 1 unit to minimize the cost.

(i) Perform three iterations of the golden section search method, using a table, to find an approximation to a local minimizer of f. State the smallest interval containing the local minimizer correct to six decimal places.

(ii) Determine the number of iterations and the number of function evaluations required by the method to reach the local minimizer correct to four decimal places. State one advantages of the golden section method over the grid search method.

(b) The manufacturing of another product within the chemical plant is found to be a function of two catalysts x_1 and x_1 as shown in the following model:

$$f(x) = 2x_1^3 + 5x_2^3 - 8x_1 - 12x_2 + 2x_1x_2 .$$

(i) Perform one iteration of the steepest descent method to find an approximation to a local minimizer of f starting from $\mathbf{x}^{(0)}=[\,0,\,0]^T$.

(ii) Find and classify all the stationary points of the function f. Estimate the absolute change in the value of the function f if the coefficient of the term $(x_2)^3$ is changed from 5 to 5.1.

Question 5

Consider the following constrained optimization model:

$$\text{maximize } f(\mathbf{x}) = -x_1^2 - 0.5x_2^2 + 8x_1 + 2x_2 + 60$$

$$\text{subject to } c(\mathbf{x}) = 40x_1 + 20x_2 - 140 = 0 .$$

(a) Apply the quadratic penalty-function method for this model to find an approximation to the local maximizer c of $f(\mathrm{x})$, with $\sigma = 10^3$.

(b) Write down the Lagrangian function, $L(\mathbf{x},\mu)$, for the model and use it to determine the local maximizer c of $f(\mathrm{x})$.

(c) Write down the augmented Lagrangian function, $\psi(\mathbf{x},\mu,\sigma)$, corresponding to this model and determine stationary point α in terms of μ and σ. Verify that the stationary point c is a local maximizer of $\psi(\mathbf{x},\mu,\sigma)$.

Set 4

Question 1

Consider the following function describing the population growth of new species over time t given as:

$$f(t) = \frac{e^t - 2}{t^4}, \qquad t \neq 0.$$

(a) A sketch of the function $f(t)$ shows that there is a maximum point at t = 0.97 and a minimum at t = 3.83. Show that the exact position of the turning points satisfy the following equations:

(i) $t = 4 - 8e^{-t}$.

(ii) $t = ln\frac{8}{4-t}$.

(b) Show that the following simple iterative scheme $t_{r+1} = g(t_r) = 4 - 8e^{-t_r}$ is a contraction mapping over the interval [3, 4] .

(c) Show that the following simple iterative scheme

$$t_{r+1} = g(t_r) = 4 - 8e^{-t_r}$$

is a contraction mapping over the interval [0, 1].

(d) Starting from $t_0 = 1$, apply a suitable iterative scheme from part (a) to find an approximation to the maximum point correct to three decimal places.

Question 2

The population growth of certain species is modelled by the following nonlinear function:

$$f(x,c) = e^{cx}\cos x - 5.$$

(a) For c = 1.0, prove that the function has one stationary point at $x = \frac{\pi}{4}$ and subsequent stationary points at $x = \frac{5\pi}{4}$ and $\frac{9\pi}{4}$.

(b) For c =1.0, use the Newton-Raphson method to find an approximation to the root of f(x) correct to one decimal place, starting from $x_0 = 4.5$.

(c) When the constant c is subject to small changes about the value c = 1.0, analyze whether or not the problem of finding the root being at $x = 4.7$ is:

(i) absolutely ill-conditioned;

(ii) relatively ill-conditioned.

(d) Use your result from (c) to estimate the absolute and relative changes to the root when c is changed from 1.0 to 1.05.

Question 3

The optimization of a two-variables chemical process requires the solution of the nonlinear set of equations given by:

$$\begin{aligned} f_1(\mathbf{x}) &= x_1 + x_2^2 - 0.4 = 0, \\ f_2(\mathbf{x}) &= x_2 - x_1^3 = 0. \end{aligned}$$

(a) To ascertain the Jacobi iterative solution scheme is guaranteed to converge. You are required to provide a Jacobi iteration scheme and show that it is a contraction mapping on the region

$$R = \{(x_1, x_2):\ 0.3 \le x_1 \le 0.5,\ \ 0.0 \le x_2 \le 0.2\}.$$

(b) Starting from $x_1^{(0)} = 0.4$, $x_2^{(0)} = 0.1$, carry out the Jacobi iteration scheme in (a) for five iterations to solve for an approximate solution, accurate to five decimal places, for the above system of equations.

(c) Write down the Gauss-Seidel iteration scheme and carry it out four iterations to find an approximate solution to three decimal places, starting from:

$$x_1^{(0)} = 0.4,\ x_2^{(0)} = 0.1.$$

(d) Comment on the convergence properties of the Jacobi and Gauss-Seidel iteration schemes in (b) and (c).

Question 4

Consider the following unconstrained objective function to be minimized on the interval [0, 1]:

$$f(x) = x^2 + e^{-x}.$$

(a) By drawing a sketch, prove that a stationary point of the function lies in the interval [0, 1]. Classify the stationary point as a local maxima, local minima or a saddle point.

(b) Starting from the interval [0, 1], use three iterations of the golden section search method to find an approximation to a local minimizer for *f*. Present your results, accurate to six decimal places in a table as shown.

r	x_0	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$
1						
2						
3						

As a result of the three iterations, state the smallest interval that contains the local minimizer.

(c) Determine the number of function evaluations and the number of iterations of the golden search method which would be required to determine a local minimizer correct to five decimal places.

(d) Determine the number of function evaluations and the number of iterations which would be required by the three interior point grid search method to achieve five decimal places accuracy. Compare briefly the performance of the golden section and the grid search methods.

Question 5

Consider the following unconstrained optimization model.

$$\text{minimize} \quad f(x_1, x_2) = 3x_1^2 - 10x_1 + 2x_2^2 - 2x_1x_2 + 12$$

(a) Perform one iteration of the alternating variable method starting from $\mathbf{x} = [0, 0]^T$ to calculate an approximation of a local minimizer of f.

(b) Carry out one iteration of the steepest descent method starting from $\mathbf{x} = [0, 0]^T$ to calculate an approximation of a local minimizer of f.

(c) Find all the stationary points of the function f and classify them as local minima, maxima or saddle point.

(d) For the minimiser found in part (c), determine the absolute condition number when the coefficient 3 of x_1^2 is subject to small changes around 3.

Question 6

Consider the following constrained optimization model.

$$\text{minimize} \quad f(\mathbf{x}) = 3x_2^2 + x_1 x_2 + 6x_1 + 7x_2$$

subject to:

$$c_1(\mathbf{x}) = 5x_1 + 3x_2 - 7 = 0$$

$$c_2(\mathbf{x}) = x_1 + x_2 - 1 = 0$$

(a) Write down the Lagrangian function for the model and demonstrate the first order necessary conditions for a constrained local minimizer for this constrained model.

(b) Find a stationary point $\alpha = [x_1, x_2]^T$ for the function f and determine the Lagrangian terms λ_1, λ_2. Show that $\alpha = [x_1, x_2]^T$ is a local minimiser.

(c) Use your results from (b) to demonstrate whether KKT conditions of optimality are satisfied.

(d) Deduce the change in the objective function if the right hand side of the constraint $c_2(x)$ is decreased by 0.2 to 6.8.

Set 5

Question 1

Consider the equation

$$f(x) = 8x^3 - e^x = 0$$

The equation has two roots in the interval [0, 9], estimated to one decimal place accuracy and given as $x_1 = 0.6$ and $x_2 = 8.5$.

(a) Show that one simple iterative scheme for solving the equation is

$$x_{r+1} = g(x_r) = 3\ln 2x_r$$

Show that this iterative scheme satisfies the condition for a contraction mapping on an interval containing the largest root.

(b) Show that the following iterative scheme for solving the above equation

$$x_{r+1} = g(x_r) = \tfrac{1}{2}e^{\frac{1}{3}x_r}.$$

will converge to the smallest root by proving the scheme is a contraction mapping on an interval containing this root.

Question 2

(a) The largest real root of the equation

$$f(x,c) = e^x - x^2 - c$$

is at $x = 1.673404$ when $c = 2.53$.

When c is subject to small changes about the value 2.53, determine whether or not the problem of finding the largest root is:

(i) absolutely ill-conditioned;

(ii) relatively ill-conditioned.

(b) Consider the system of linear equations A.x = b

$$\begin{bmatrix} 1 & -2 & -3 \\ 4 & -5 & -11 \\ -3 & 12 & 8 \end{bmatrix} \cdot \mathbf{x} = \begin{bmatrix} 3 \\ 14 \\ -2 \end{bmatrix}. \quad \text{with } \mathbf{A}^{-1} = \begin{bmatrix} -10.222 & 2.222 & -0.778 \\ -0.111 & 0.111 & 0.111 \\ -3.667 & 0.667 & -0.333 \end{bmatrix}$$

Determine the absolute condition number and an upper bound of the relative condition number for the system with respect to small changes in the right-hand side of the equations.

Question 3

Consider the following system of nonlinear equations.

$$f_1(\mathbf{x}) = x_1 x_2 - 1 \quad = 0$$

$$f_2(\mathbf{x}) = 0.5 - x_2^2 + 2x_2 - x_1 \quad = 0$$

(a) Write down the Jacobi iterative scheme for this system of equations in the order given. Show that it is a contraction mapping on the region

$$\mathrm{R} = \{(x_1, x_2) : \ 0.6 \le x_1 \le 1, \ \ 1.2 \le x_2 \le 1.6\}.$$

(b) Write down the Gauss-Seidel iterative scheme and carry out three iterations, starting from $x_1^{(0)} = 1.4$, $x_2^{(0)} = 0.7$ and using 5 decimal places accuracy.

Question 4

Consider the following constrained minimization model.

minimize $f(\mathbf{x}) = 10x_1^3 + 30x_1^2 + 3x_2^2 + 6x_1 x_2 - 5x_3^2 + 60x_1 - 4x_2 + 8$

subject to

$$c_1(\mathbf{x}) = 3x_1 - x_2 - 12 = 0$$

$$c_2(\mathbf{x}) = 6x_1 + x_3 - 10 = 0$$

(a) Write down the Lagrangian function for the model, and use it to determine the first-order necessary conditions for a constrained local minimizer for the model.

Hence show that $\alpha = [3, \ -3, \ -8]^{\mathrm{T}}$ is a constrained stationary point of the model and determine the Lagrange multipliers.

(b) Estimate the change in the minimum value of f if the second constraint constant term is changed from 10 to 10.1.

Question 5

Consider the function

$$f(x_1, x_2) = 3x_1^2 + x_2^2 + 2x_1x_2 - 4x_2$$

(a) Find and classify all the stationary points of the function. f.

(b) Perform two iterations of the steepest descent method, starting from the origin $\mathbf{x}^{(0)} = [0, 0]^T$, to find an approximation to a local minimizer of f.

Question 6

Consider the function

$$f(x_1, x_2) = 3x_1^2 + 3x_2^2 + 3x_1x_2 - 13x_1 + 2x_2 + 1$$

(a) Find and classify all the stationary points of the function. f.

(b) Perform two iterations of the alternating variables method, starting from the origin, $\mathbf{x}^{(0)} = [0, 0]^T$, to find an approximation to a local minimizer of f.

Question 7

Consider the following constrained minimization model.

$$\text{minimize } f(\mathbf{x}) = x_1^2 - 3x_1 x_2 + 2x_2^2 - x_1 + 13x_2$$

$$\text{subject to } c(\mathbf{x}) = x_1 + 4x_2 - 7 = 0.$$

(a) Determine the value of α_2 such that $\boldsymbol{\alpha} = [3, \alpha_2]^T$ is a feasible point.

(b) Determine the feasible direction S, ($S \neq 0$), at the above point

(c) Write down the Lagrangian function for the model and use it to determine the first order necessary conditions for $\boldsymbol{\alpha}$ to be a constrained local minimizer. Deduce the Lagrange multiplier λ.

Set 6

Question 1

Consider the equation

$$f(x) = 4x^2 - e^x = 0$$

The equation has two positive roots in the interval [0, 5], estimated to one decimal place accuracy as $x_1 = 0.7$ and $x_2 = 4.3$.

(a) Show that one simple iterative scheme for solving the equation is

$$x_{r+1} = g(x_r) = 2\ln 2x_r$$

Prove that this iterative scheme satisfies the condition for a contraction mapping on an interval containing the largest root.

(b) Show that the following simple iterative scheme

$$x_{r+1} = g(x_r) = \tfrac{1}{2}e^{\frac{1}{2}x_r}$$

is a contraction mapping over an interval containing the smallest root in the interval [0, 5].

Question 2

(a) The largest real root of the equation

$$f(x,c) = \cos 2x - x^2 - c$$

is at $x = 0.514852$ when $c = 0.25$.

When c is subject to small changes about the value 0.25, determine whether or not the problem of finding the largest root is:
(i) absolutely ill-conditioned;
(ii) relatively ill-conditioned.

(b) Consider the system of linear equations **A.x** = **b.**

$$\begin{bmatrix} 1 & -2 & -3 \\ 4 & -5 & -11 \\ -3 & 12 & 8 \end{bmatrix} \cdot \mathbf{x} = \begin{bmatrix} 3 \\ 14 \\ -2 \end{bmatrix}. \quad \text{with } \mathbf{A}^{-1} = \begin{bmatrix} -10.222 & 2.222 & -0.778 \\ -0.111 & 0.111 & 0.111 \\ -3.667 & 0.667 & -0.333 \end{bmatrix}$$

Determine the absolute condition number and an upper bound of the relative condition number for the system with respect to small changes in the right-hand side of the equations.

Question 3

Consider the following system of nonlinear equations.

$$f_1(\mathbf{x}) = 0.5 - x_1^2 + 2x_1 - x_2 \quad = 0$$
$$f_2(\mathbf{x}) = x_1 x_2 - 1 \quad = 0$$

(a) Write down the Jacobi iterative scheme for this system of equations in the order given. Show that it is a contraction mapping on the region

$$R = \{(x_1, x_2):\ 0.6 \le x_1 \le 0.9,\ \ 1.2 \le x_2 \le 1.5\}.$$

(b) Write down the Gauss-Seidel iterative scheme and carry out three iterations, starting from $x_1^{(0)} = 0.7$, $x_2^{(0)} = 1.4$ and using 5 decimal places accuracy.

Question 4

Consider the following constrained minimization model.

$$\text{minimize} \quad f(\mathbf{x}) = 2x_1^2 + 8x_2^2 - 3x_1 x_2 - 5x_1 + 9x_2$$
$$\text{subject to} \quad c(\mathrm{x}) = x_1 + 4x_2 - 7 = 0.$$

(a) Write down the Lagrangian function for the model.

(b) Show that the point $\alpha = [3,\ 1]^T$ is a constrained stationary point of *f*. Determine the Lagrangian term λ.

(c) Estimate the change in the minimum value of *f* if the constraint constant term is changed from 7 to 7.1.

Question 5

Consider the function

$$f(x_1, x_2) = x_1^2 + 8x_2^2 + 2x_1x_2 + 2x_2 .$$

(a) Find and classify all the stationary points of the function. f.

(b) Perform two iterations of the steepest descent method, starting from the origin $\mathbf{x}^{(0)} = [0, 0]^{\mathrm{T}}$, to find an approximation to a local minimizer of f.

Question 6

Consider the function

$$f(x_1, x_2) = 2x_1^2 + 2x_2^2 + 3x_1x_2 + 13x_1 - 2x_2 + 1 .$$

(a) Find and classify all the stationary points of the function. f.

(b) Perform two iterations of the alternating variables method, starting from the origin, $\mathbf{x}^{(0)} = [0, 0]^{\mathrm{T}}$, to find an approximation to a local minimizer of f.

Question 7

Consider the following constrained minimization model

$$\text{minimize} \quad f(\mathrm{x}) = x_1^2 - 3x_1x_2 + 2x_2^2 - x_1 + 13x_2$$

$$\text{subject to} \quad c(\mathrm{x}) = x_1 + 4x_2 - 7 = 0 .$$

(a) Write down the Lagrangian function for this model and use it to determine the first order necessary conditions. Deduce the stationary point α and the Lagrange multiplier λ.

(b) Show that α is a constrained local minimizer for the model.