we conclude that p divides both m and n, contradicting the choice of m and n as relatively prime.

Since m^2+n^2 is relatively prime to m^2-n^2, and m^2+n^2 divides the product $(m^2-n^2)c$, we know that m^2+n^2 divides c.

Then the integer $c/(m^2+n^2)$ divides both a and b, which are relatively prime. This means that $c/(m^2+n^2)=1$, and we have shown that $a=m^2-n^2$, $b=2mn$, and $c=m^2+n^2$ as desired. □

Remark: It might be instructive to follow the proof to find the values of m and n in a particular case. Consider the reduced triple 48, 55, 73. The corresponding norm 1 element is $\beta = 55/73+48i/73$. Set $\alpha = 1+\beta$. Then the quotient

$$\begin{aligned}
\beta &= \alpha/\overline{\alpha} \\
&= \frac{1+55/73+48i/73}{1+55/73-48i/73} && \text{clear denominators} \\
&= \frac{73+55+48i}{73+55-48i} \\
&= \frac{128+48i}{128-48i} && \text{cancel common factors} \\
&= \frac{8+3i}{8-3i}
\end{aligned}$$

gives the parametrization $m=8$ and $n=3$ for this triple.

Exercises:

1. Check that the converse of the theorem holds. Suppose m and n are a pair of relatively prime positive integers of opposite parity, with $m>n$. Show that the formulas $a=m^2-n^2$, $b=2mn$, and $c=m^2+n^2$ define a reduced Pythagorean triple.

2. Show that the integers m and n in the theorem are uniquely determined by the reduced Pythagorean triple.

3. Find the m and n which parametrize the reduced Pythagorean triple 175, 288, 337.

4. Find the m and n which parametrize the reduced Pythagorean triple 120, 209, 241.

Project: Study the arithmetic in the Gaussian integers $\mathbb{Z}[i]=\{a+bi|a,b\in\mathbb{Z}\}$ to give a different derivation of the classification as follows.

Develop a division algorithm in $\mathbb{Z}[i]$ and show that, up to units, unique factorization into irreducibles (primes) holds in $\mathbb{Z}[i]$.

An element u in $\mathbb{Z}[i]$ is a unit if, and only if, its norm is a unit in the rational integers $\mathbb{Z}$, i.e., $u\overline{u}=\pm1$. The four units in $\mathbb{Z}[i]$ are $\pm1, \pm i$.

Suppose a,b,c is a reduced Pythagorean triple. Consider the factorization $c^2 = a^2+b^2=(a+bi)(a-bi)$ in the Gaussian integers. We have shown that c is odd.