1. A couple expects a baby, who is equally likely to be a boy or a girl. An ultrasound scan can either reveal the baby's true gender, or can fail to say anything. For a boy, the probability of discovery is $1/2$, while for a girl it is $1/4$. Find:

   (a) the probability of a baby girl, if one scan fails;

   (b) the probability of a baby girl, if 2 scans fail;

   (c) the probability of a baby girl, if n scans fail;
   (Hint: call x the probability of a girl, and see how it changes if a scan fails. Iterate the procedure a few times, and guess the general formula. Then check it by induction.)

   (d) the distribution of the number of scans necessary to reveal the baby's gender, before performing any scan.

2. For a positive random variable X with density $f_X(t)$, the *hazard rate* is defined as the function $h_X(x) = f_X(x)/(1 - F_X(x))$. Show that:

   (a) Show that $P(X > x) = e^{-\int_0^x h_X(u)du}$.

   (b) Prove that $\int_0^\infty h_X(x)dx = \infty$.

   (c) For X, Y independent, and $Z = \min(X, Y)$, show that $h_Z(t) = h_X(t) + h_Y(t)$.

   (d) If $h_X(x) = \lambda x$, find $f_X(x)$.

   (e) For X uniform on $[0, 1]$, find $h_X(x)$.

3. Let X_1 be exponential with parameter λ, X_2 exponential with parameter $1/X_1$, X_3 exponential with parameter $1/X_2$, and so on. Find:

   (a) $E[X_n]$ for any n.

   (b) $\mathrm{Var}(X_n)$ for any n.

   (c) $\mathrm{Cov}(X_n, X_m)$ for $m < n$.

   (d) $\mathrm{Var}(X_1 + \cdots + X_n)$

Solve the following problems, supporting your answers clearly.

1. It is raining, and you are waiting for a bus, which arrives at a time U, uniform on $[0, 1]$ (in hours). There is also the possibility that a lightning arrives, according to an independent exponential time with rate λ. find:
   (a) the probability that the bus arrives before the lightning;
   (b) the probability that the bus arrives before the lightning, if neither has arrived for half an hour;
   (c) the conditional density of U with respect of the event $T < U$ (lighting arrives before bus)

2. Two lazy fund managers randomly pick 5 stocks out of 15 stocks avaiable. Then, each manager forms a portfolio with equal weights in the 5 picked stocks. The returns on the 15 stocks have IID standard normal. Find:
   (a) the distribution of C, the number of stocks picked by both managers;
   (b) the correlation between the two portfolios returns, conditionally on C;
   (c) the correlation between the two portfolios returns.

3. Let $T \sim \Gamma(2, \lambda)$ and, conditionally on T, let X be exponential with parameter T. Find:
   (a) the marginal density of X;
   (b) the density of T, conditional on X;
   (c) the probability that $X > T$.