(b) Find the rate of change in the number of Americans who are expected to be over 100 years old in the year 2015.

SOLUTION The year 2015 corresponds to $t = 15$.

$$f'(15) = 0.02829(15)^2 - 0.940(15) + 11.085 = 3.35025$$

The number of Americans over 100 years old is expected to grow at a rate of about 3.35 thousand, or about 3350, per year in the year 2015.

12.1 EXERCISES

Find the derivative of each function defined as follows.

1. $y = 12x^3 - 8x^2 + 7x + 5$
2. $y = 8x^3 - 5x^2 - \dfrac{x}{12}$
3. $y = 3x^4 - 6x^3 + \dfrac{x^2}{8} + 5$
4. $y = 5x^4 + 9x^3 + 12x^2 - 7x$
5. $f(x) = 6x^{3.5} - 10x^{0.5}$
6. $f(x) = -2x^{1.5} + 12x^{0.5}$
7. $y = 8\sqrt{x} + 6x^{3/4}$
8. $y = -100\sqrt{x} - 11x^{2/3}$
9. $y = 10x^{-3} + 5x^{-4} - 8x$
10. $y = 5x^{-5} - 6x^{-2} + 13x^{-1}$
11. $f(t) = \dfrac{7}{t} - \dfrac{5}{t^3}$
12. $f(t) = \dfrac{14}{t} + \dfrac{12}{t^4} + \sqrt{2}$
13. $y = \dfrac{6}{x^4} + \dfrac{7}{x^3} + \dfrac{3}{x} + \sqrt{5}$
14. $y = \dfrac{3}{x^6} + \dfrac{1}{x^5} - \dfrac{7}{x^2}$
15. $p(x) = -10x^{-1/2} + 8x^{-3/2}$
16. $h(x) = x^{-1/2} - 14x^{-3/2}$
17. $y = \dfrac{6}{\sqrt[4]{x}}$
18. $y = \dfrac{-2}{\sqrt[3]{x}}$
19. $f(x) = \dfrac{x^3 + 5}{x}$
20. $g(x) = \dfrac{x^3 - 4x}{\sqrt{x}}$
21. $g(x) = (8x^2 - 4x)^2$
22. $h(x) = (x^2 - 1)^3$
23. Which of the following describes the derivative function $f'(x)$ of a quadratic function $f(x)$?
    a. Quadratic **b.** Linear **c.** Constant **d.** Cubic (third degree)
24. Which of the following describes the derivative function $f'(x)$ of a cubic (third degree) function $f(x)$?
    a. Quadratic **b.** Linear **c.** Constant **d.** Cubic
25. Explain the relationship between the slope and the derivative of $f(x)$ at $x = a$.
26. Which of the following do *not* equal $\dfrac{d}{dx}(4x^3 - 6x^{-2})$?
    a. $\dfrac{12x^2 + 12}{x^3}$ **b.** $\dfrac{12x^5 + 12}{x^3}$ **c.** $12x^2 + \dfrac{12}{x^3}$
    d. $12x^3 + 12x^{-3}$

Find each derivative.

27. $D_x\left[9x^{-1/2} + \dfrac{2}{x^{3/2}}\right]$
28. $D_x\left[\dfrac{8}{\sqrt[4]{x}} - \dfrac{3}{\sqrt{x^3}}\right]$
29. $f'(-2)$ if $f(x) = \dfrac{x^4}{6} - 3x$
30. $f'(3)$ if $f(x) = \dfrac{x^3}{9} - 7x^2$

In Exercises 31–34, find the slope of the tangent line to the graph of the given function at the given value of x. Find the equation of the tangent line in Exercises 31 and 32.

31. $y = x^4 - 5x^3 + 2;\quad x = 2$
32. $y = -3x^5 - 8x^3 + 4x^2;\quad x = 1$
33. $y = -2x^{1/2} + x^{3/2};\quad x = 9$
34. $y = -x^{-3} + x^{-2};\quad x = 2$
35. Find all points on the graph of $f(x) = 9x^2 - 8x + 4$ where the slope of the tangent line is 0.
36. Find all points on the graph of $f(x) = x^3 + 9x^2 + 19x - 10$ where the slope of the tangent line is -5.

In Exercises 37–40, for each function find all values of x where the tangent line is horizontal.

37. $f(x) = 2x^3 + 9x^2 - 60x + 4$

38. $f(x) = x^3 + 15x^2 + 63x - 10$

39. $f(x) = x^3 - 4x^2 - 7x + 8$

40. $f(x) = x^3 - 5x^2 + 6x + 3$

41. At what points on the graph of $f(x) = 6x^2 + 4x - 9$ is the slope of the tangent line -2?

42. At what points on the graph of $f(x) = 2x^3 - 9x^2 - 12x + 5$ is the slope of the tangent line 12?

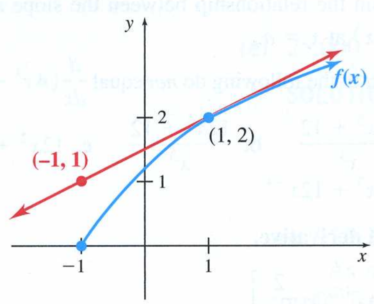
43. At what points on the graph of $f(x) = x^3 + 6x^2 + 21x + 2$ is the slope of the tangent line 9?

44. If $g'(5) = 12$ and $h'(5) = -3$, find $f'(5)$ for $f(x) = 3g(x) - 2h(x) + 3$.

45. If $g'(2) = 7$ and $h'(2) = 14$, find $f'(2)$ for $f(x) = \frac{1}{2}g(x) + \frac{1}{4}h(x)$.

46. Use the information given in the figure to find the following values.

a. $f(1)$ **b.** $f'(1)$ **c.** The domain of f **d.** The range of f



47. Explain the concept of marginal cost. How does it relate to cost? How is it found?

48. In Exercises 43–46 of Section 10.2, the effect of a when graphing $y = af(x)$ was discussed. Now describe how this relates to the fact that $D_x[af(x)] = af'(x)$.

49. Show that, for any constant k,

$$\frac{d}{dx}\left[\frac{f(x)}{k}\right] = \frac{f'(x)}{k}.$$

50. Use the differentiation feature on your graphing calculator to solve the problems (to 2 decimal places) below, where $f(x)$ is defined as follows:

$$f(x) = 1.25x^3 + 0.01x^2 - 2.9x + 1.$$

a. Find $f'(4)$.

b. Find all values of x where $f'(x) = 0$.

APPLICATIONS

Business and Economics

51. Revenue Assume that a demand equation is given by $q = 5000 - 100p$. Find the marginal revenue for the following production levels (values of q). (*Hint:* Solve the demand equation for p and use $R(q) = qp$.)

a. 1000 units **b.** 2500 units **c.** 3000 units

52. Profit Suppose that for the situation in Exercise 51 the cost of producing q units is given by $C(q) = 3000 - 20q + 0.03q^2$. Find the marginal profit for the following production levels.

a. 500 units **b.** 815 units **c.** 1000 units

53. Revenue If the price in dollars of a stereo system is given by

$$p(q) = \frac{1000}{q^2} + 1000,$$

where q represents the demand for the product, find the marginal revenue when the demand is 10.

54. Profit Suppose that for the situation in Exercise 53 the cost in dollars of producing q stereo systems is given by $C(q) = 0.2q^2 + 6q + 50$. Find the marginal profit when the demand is 10.

55. Sales Often sales of a new product grow rapidly at first and then level off with time. This is the case with the sales represented by the function

$$S(t) = 100 - 100t^{-1},$$

where t represents time in years. Find the rate of change of sales for the following numbers of years.

a. 1 **b.** 10

56. Profit An analyst has found that a company's costs and revenues in dollars for one product are given by

$$C(x) = 2x \quad \text{and} \quad R(x) = 6x - \frac{x^2}{1000},$$

respectively, where x is the number of items produced.

a. Find the marginal cost function.

b. Find the marginal revenue function.

c. Using the fact that profit is the difference between revenue and costs, find the marginal profit function.

d. What value of x makes marginal profit equal 0?

e. Find the profit when the marginal profit is 0.

(As we shall see in the next chapter, this process is used to find *maximum* profit.)

57. Postal Rates U.S. postal rates have steadily increased since 1932. Using data depicted in the table for the years 1932–2009, the cost in cents to mail a single letter can be modeled using a quadratic formula as follows:

$$C(t) = 0.008446t^2 - 0.08924t + 1.254$$

where t is the number of years since 1932. ***Source:** U.S. Postal Service.*

Year	Cost	Year	Cost
1932	3	1988	25
1958	4	1991	29
1963	5	1995	32
1968	6	1999	33
1971	8	2001	34
1974	10	2002	37
1975	13	2006	39
1978	15	2007	41
1981	18	2008	42
1981	20	2009	44
1985	22		

a. Find the predicted cost of mailing a letter in 1982 and 2002 and compare these estimates with the actual rates.

b. Find the rate of change of the postage cost for the years 1982 and 2002 and interpret your results.

c. Using the regression feature on a graphing calculator, find a cubic function that models this data, letting $t = 0$ correspond to the year 1932. Then use your answer to find the rate of change of the postage cost for the years 1982 and 2002.

d. Discuss whether the quadratic or cubic function best describes the data. Do the answers from part b or from part c best describe the rate that postage was going up in the years 1982 and 2002?

e. Explore other functions that could be used to model the data, using the various regression features on a graphing calculator, and discuss to what extent any of them are useful descriptions of the data.

58. Money The total amount of money in circulation for the years 1950–2009 can be closely approximated by

$$M(t) = 0.005209t^3 - 0.04159t^2 - 0.3664t + 34.49$$

where t represents the number of years since 1950 and $M(t)$ is in billions of dollars. Find the derivative of $M(t)$ and use it to find the rate of change of money in circulation in the following years. *Source: U.S. Treasury.*

a. 1960 **b.** 1980 **c.** 1990 **d.** 2000

e. What do your answers to parts a–d tell you about the amount of money in circulation in those years?

Life Sciences

59. Cancer Insulation workers who were exposed to asbestos and employed before 1960 experienced an increased likelihood of lung cancer. If a group of insulation workers has a cumulative total of 100,000 years of work experience with their first date of employment t years ago, then the number of lung cancer cases occurring within the group can be modeled using the function

$$N(t) = 0.00437t^{3.2}.$$

Find the rate of growth of the number of workers with lung cancer in a group as described by the following first dates of employment. *Source: Observation and Inference: An Introduction to the Methods of Epidemiology.*

a. 5 years ago **b.** 10 years ago

60. Blood Sugar Level Insulin affects the glucose, or blood sugar, level of some diabetics according to the function

$$G(x) = -0.2x^2 + 450,$$

where $G(x)$ is the blood sugar level 1 hour after x units of insulin are injected. (This mathematical model is only approximate, and it is valid only for values of x less than about 40.) Find the blood sugar level after the following numbers of units of insulin are injected.

a. 0 **b.** 25

Find the rate of change of blood sugar level after injection of the following numbers of units of insulin.

c. 10 **d.** 25

61. Bighorn Sheep The cumulative horn volume for certain types of bighorn rams, found in the Rocky Mountains, can be described by the quadratic function

$$V(t) = -2159 + 1313t - 60.82t^2,$$

where $V(t)$ is the horn volume (in cm^3) and t is the year of growth, $2 \le t \le 9$. *Source: Conservation Biology.*

a. Find the horn volume for a 3-year-old ram.

b. Find the rate at which the horn volume of a 3-year-old ram is changing.

62. Brain Mass The brain mass of a human fetus during the last trimester can be accurately estimated from the circumference of the head by

$$m(c) = \frac{c^3}{100} - \frac{1500}{c}$$

where $m(c)$ is the mass of the brain (in grams) and c is the circumference (in centimeters) of the head. *Source: Early Human Development.*

a. Estimate the brain mass of a fetus that has a head circumference of 30 cm.

b. Find the rate of change of the brain mass for a fetus that has a head circumference of 30 cm and interpret your results.

63. Velocity of Marine Organism The typical velocity (in centimeters per second) of a marine organism of length l (in centimeters) is given by $v = 2.69l^{1.86}$. Find the rate of change of the velocity with respect to the length of the organism. *Source: Mathematical Topics in Population Biology Morphogenesis and Neurosciences.*

64. Heart The left ventricular length (viewed from the front of the heart) of a fetus that is at least 18 weeks old can be estimated by

$$l(x) = -2.318 + 0.2356x - 0.002674x^2,$$

where $l(x)$ is the ventricular length (in centimeters) and x is the age (in weeks) of the fetus. *Source: American Journal of Cardiology.*

a. Determine a meaningful domain for this function.

b. Find $l'(x)$.

c. Find $l'(25)$.

12.2 EXERCISES

Use the product rule to find the derivative of the following. (Hint for Exercises 3–6: Write the quantity as a product.)

1. $y = (3x^2 + 2)(2x - 1)$
2. $y = (5x^2 - 1)(4x + 3)$
3. $y = (2x - 5)^2$
4. $y = (7x - 6)^2$
5. $k(t) = (t^2 - 1)^2$
6. $g(t) = (3t^2 + 2)^2$
7. $y = (x + 1)(\sqrt{x} + 2)$
8. $y = (2x - 3)(\sqrt{x} - 1)$
9. $p(y) = (y^{-1} + y^{-2})(2y^{-3} - 5y^{-4})$
10. $q(x) = (x^{-2} - x^{-3})(3x^{-1} + 4x^{-4})$

Use the quotient rule to find the derivative of the following.

11. $f(x) = \dfrac{6x + 1}{3x + 10}$
12. $f(x) = \dfrac{8x - 11}{7x + 3}$
13. $y = \dfrac{5 - 3t}{4 + t}$
14. $y = \dfrac{9 - 7t}{1 - t}$
15. $y = \dfrac{x^2 + x}{x - 1}$
16. $y = \dfrac{x^2 - 4x}{x + 3}$
17. $f(t) = \dfrac{4t^2 + 11}{t^2 + 3}$
18. $y = \dfrac{-x^2 + 8x}{4x^2 - 5}$
19. $g(x) = \dfrac{x^2 - 4x + 2}{x^2 + 3}$
20. $k(x) = \dfrac{x^2 + 7x - 2}{x^2 - 2}$
21. $p(t) = \dfrac{\sqrt{t}}{t - 1}$
22. $r(t) = \dfrac{\sqrt{t}}{2t + 3}$
23. $y = \dfrac{5x + 6}{\sqrt{x}}$
24. $y = \dfrac{4x - 3}{\sqrt{x}}$
25. $h(z) = \dfrac{z^{2.2}}{z^{3.2} + 5}$
26. $g(y) = \dfrac{y^{1.4} + 1}{y^{2.5} + 2}$
27. $f(x) = \dfrac{(3x^2 + 1)(2x - 1)}{5x + 4}$
28. $g(x) = \dfrac{(2x^2 + 3)(5x + 2)}{6x - 7}$
29. If $g(3) = 4$, $g'(3) = 5$, $f(3) = 9$, and $f'(3) = 8$, find $h'(3)$ when $h(x) = f(x)g(x)$.
30. If $g(3) = 4$, $g'(3) = 5$, $f(3) = 9$, and $f'(3) = 8$, find $h'(3)$ when $h(x) = f(x)/g(x)$.
31. Find the error in the following work.

$$D_x\left(\frac{2x + 5}{x^2 - 1}\right) = \frac{(2x + 5)(2x) - (x^2 - 1)2}{(x^2 - 1)^2} = \frac{4x^2 + 10x - 2x^2 + 2}{(x^2 - 1)^2} = \frac{2x^2 + 10x + 2}{(x^2 - 1)^2}$$

32. Find the error in the following work.

$$D_x\left(\frac{x^2 - 4}{x^3}\right) = x^3(2x) - (x^2 - 4)(3x^2) = 2x^4 - 3x^4 + 12x^2 = -x^4 + 12x^2$$

33. Find an equation of the line tangent to the graph of $f(x) = x/(x - 2)$ at $(3, 3)$.
34. Find an equation of the line tangent to the graph of $f(x) = (2x - 1)(x + 4)$ at $(1, 5)$.
35. Consider the function

$$f(x) = \frac{3x^3 + 6}{x^{2/3}}.$$

   a. Find the derivative using the quotient rule.

   b. Find the derivative by first simplifying the function to

$$f(x) = \frac{3x^3}{x^{2/3}} + \frac{6}{x^{2/3}} = 3x^{7/3} + 6x^{-2/3}$$

   and using the rules from the previous section.

   c. Compare your answers from parts a and b and explain any discrepancies.

36. What is the result of applying the product rule to the function

$$f(x) = kg(x),$$

where k is a constant? Compare with the rule for differentiating a constant times a function from the previous section.

37. Following the steps used to prove the product rule for derivatives, prove the quotient rule for derivatives.
38. Use the fact that $f(x) = u(x)/v(x)$ can be rewritten as $f(x)v(x) = u(x)$ and the product rule for derivatives to verify the quotient rule for derivatives. (*Hint:* After applying the product rule, substitute $u(x)/v(x)$ for $f(x)$ and simplify.)

For each function, find the value(s) of x in which $f'(x) = 0$, to 3 decimal places.

39. $f(x) = (x^2 - 2)(x^2 - \sqrt{2})$

40. $f(x) = \dfrac{x - 2}{x^2 + 4}$

APPLICATIONS

Business and Economics

41. **Average Cost** The total cost (in hundreds of dollars) to produce x units of perfume is

$$C(x) = \frac{3x + 2}{x + 4}.$$

Find the average cost for each production level.

a. 10 units **b.** 20 units **c.** x units

d. Find the marginal average cost function.

42. **Average Profit** The total profit (in tens of dollars) from selling x self-help books is

$$P(x) = \frac{5x - 6}{2x + 3}.$$

Find the average profit from each sales level.

a. 8 books **b.** 15 books **c.** x books

d. Find the marginal average profit function.

e. Is this a reasonable function for profit? Why or why not?

43. **Employee Training** A company that manufactures bicycles has determined that a new employee can assemble $M(d)$ bicycles per day after d days of on-the-job training, where

$$M(d) = \frac{100d^2}{3d^2 + 10}.$$

a. Find the rate of change function for the number of bicycles assembled with respect to time.

b. Find and interpret $M'(2)$ and $M'(5)$.

44. **Marginal Revenue** Suppose that the demand function is given by $p = D(q)$, where q is the quantity that consumers demand when the price is p. Show that the marginal revenue is given by

$$R'(q) = D(q) + qD'(q).$$

45. **Marginal Average Cost** Suppose that the average cost function is given by $\overline{C}(x) = C(x)/x$, where x is the number of items produced. Show that the marginal average cost function is given by

$$\overline{C}'(x) = \frac{xC'(x) - C(x)}{x^2}.$$

46. **Revenue** Suppose that at the beginning of the year, a Vermont maple syrup distributor found that the demand for maple syrup, sold at \$15 a quart, was 500 quarts each month. At that time, the price was going up at a rate of \$0.50 a month, but despite this, the demand was going up at a rate of 30 quarts a month due to increased advertising. How fast was the revenue increasing?

47. **Average Cost** A gasoline refinery found that the cost to produce 12,500 gallons of gasoline last month was \$27,000. At that time, the cost was going up at a rate of \$1200 per month, while the number of gallons of gasoline the refinery produced was going up at a rate of 350 gallons per month. At what rate was the average cost to produce a gallon of gasoline increasing or decreasing last month?

Life Sciences

48. **Muscle Reaction** When a certain drug is injected into a muscle, the muscle responds by contracting. The amount of contraction, s (in millimeters) is related to the concentration of the drug, x (in milliliters) by

$$s(x) = \frac{x}{m + nx},$$

where m and n are constants.

a. Find $s'(x)$.

b. Find the rate of contraction when the concentration of the drug is 50 ml, $m = 10$, and $n = 3$.

49. **Growth Models** In Exercise 58 of Section 10.3, the formula for the growth rate of a population in the presence of a quantity x of food was given as

$$f(x) = \frac{Kx}{A + x}.$$

This was referred to as Michaelis-Menten kinetics.

a. Find the rate of change of the growth rate with respect to the amount of food.

b. The quantity A in the formula for $f(x)$ represents the quantity of food for which the growth rate is half of its maximum. Using your answer from part a, find the rate of change of the growth rate when $x = A$.

50. **Bacteria Population** Assume that the total number (in millions) of bacteria present in a culture at a certain time t (in hours) is given by

$$N(t) = 3t(t - 10)^2 + 40.$$

a. Find $N'(t)$.

Find the rate at which the population of bacteria is changing at the following times.

b. 8 hours **c.** 11 hours

d. The answer in part b is negative, and the answer in part c is positive. What does this mean in terms of the population of bacteria?

51. **Work/Rest Cycles** Murrell's formula for calculating the total amount of rest, in minutes, required after performing a particular type of work activity for 30 minutes is given by the formula

$$R(w) = \frac{30(w - 4)}{w - 1.5},$$

where w is the work expended in kilocalories per minute, kcal/min. *Source: Human Factors in Engineering and Design.*

a. A value of 5 for w indicates light work, such as riding a bicycle on a flat surface at 10 mph. Find $R(5)$.

b. A value of 7 for w indicates moderate work, such as mowing grass with a pushmower on level ground. Find $R(7)$.

c. Find $R'(5)$ and $R'(7)$ and compare your answers. Explain whether these answers make sense.

12.3 EXERCISES

Let $f(x) = 5x^2 - 2x$ and $g(x) = 8x + 3$. Find the following.

1. $f[g(2)]$ **2.** $f[g(-5)]$ **3.** $g[f(2)]$

4. $g[f(-5)]$ **5.** $f[g(k)]$ **6.** $g[f(5z)]$

In Exercises 7–14, find $f[g(x)]$ and $g[f(x)]$.

7. $f(x) = \dfrac{x}{8} + 7;\quad g(x) = 6x - 1$

8. $f(x) = -8x + 9;\quad g(x) = \dfrac{x}{5} + 4$

9. $f(x) = \dfrac{1}{x};\quad g(x) = x^2$

10. $f(x) = \dfrac{2}{x^4};\quad g(x) = 2 - x$

11. $f(x) = \sqrt{x + 2};\quad g(x) = 8x^2 - 6$

12. $f(x) = 9x^2 - 11x;\quad g(x) = 2\sqrt{x + 2}$

13. $f(x) = \sqrt{x + 1};\quad g(x) = \dfrac{-1}{x}$

14. $f(x) = \dfrac{8}{x};\quad g(x) = \sqrt{3 - x}$

Write each function as the composition of two functions. (There may be more than one way to do this.)

15. $y = (5 - x^2)^{3/5}$ **16.** $y = (3x^2 - 7)^{2/3}$

17. $y = -\sqrt{13 + 7x}$ **18.** $y = \sqrt{9 - 4x}$

19. $y = (x^2 + 5x)^{1/3} - 2(x^2 + 5x)^{2/3} + 7$

20. $y = (x^{1/2} - 3)^2 + (x^{1/2} - 3) + 5$

Find the derivative of each function defined as follows.

21. $y = (8x^4 - 5x^2 + 1)^4$ **22.** $y = (2x^3 + 9x)^5$

23. $k(x) = -2(12x^2 + 5)^{-6}$ **24.** $f(x) = -7(3x^4 + 2)^{-4}$

25. $s(t) = 45(3t^3 - 8)^{3/2}$ **26.** $s(t) = 12(2t^4 + 5)^{3/2}$

27. $g(t) = -3\sqrt{7t^3 - 1}$ **28.** $f(t) = 8\sqrt{4t^2 + 7}$

29. $m(t) = -6t(5t^4 - 1)^4$ **30.** $r(t) = 4t(2t^5 + 3)^4$

31. $y = (3x^4 + 1)^4(x^3 + 4)$ **32.** $y = (x^3 + 2)(x^2 - 1)^4$

33. $q(y) = 4y^2(y^2 + 1)^{5/4}$ **34.** $p(z) = z(6z + 1)^{4/3}$

35. $y = \dfrac{-5}{(2x^3 + 1)^2}$ **36.** $y = \dfrac{1}{(3x^2 - 4)^5}$

37. $r(t) = \dfrac{(5t - 6)^4}{3t^2 + 4}$ **38.** $p(t) = \dfrac{(2t + 3)^3}{4t^2 - 1}$

39. $y = \dfrac{3x^2 - x}{(2x - 1)^5}$ **40.** $y = \dfrac{x^2 + 4x}{(3x^3 + 2)^4}$

41. In your own words explain how to form the composition of two functions.

42. The generalized power rule says that if $g(x)$ is a function of x and $y = [g(x)]^n$ for any real number n, then

$$\frac{dy}{dx} = n \cdot [g(x)]^{n-1} \cdot g'(x).$$

Explain why the generalized power rule is a consequence of the chain rule and the power rule.

Consider the following table of values of the functions f and g and their derivatives at various points.

x	1	2	3	4
$f(x)$	2	4	1	3
$f'(x)$	−6	−7	−8	−9
$g(x)$	2	3	4	1
$g'(x)$	2/7	3/7	4/7	5/7

Find the following using the table above.

43. a. $D_x(f[g(x)])$ at $x = 1$ **b.** $D_x(f[g(x)])$ at $x = 2$

44. a. $D_x(g[f(x)])$ at $x = 1$ **b.** $D_x(g[f(x)])$ at $x = 2$

In Exercises 45–48, find the equation of the tangent line to the graph of the given function at the given value of x.

45. $f(x) = \sqrt{x^2 + 16};\quad x = 3$

46. $f(x) = (x^3 + 7)^{2/3};\quad x = 1$

47. $f(x) = x(x^2 - 4x + 5)^4;\quad x = 2$

48. $f(x) = x^2\sqrt{x^4 - 12};\quad x = 2$

In Exercises 49 and 50, find all values of x for the given function where the tangent line is horizontal.

49. $f(x) = \sqrt{x^3 - 6x^2 + 9x + 1}$

50. $f(x) = \dfrac{x}{(x^2 + 4)^4}$

51. Katie and Sarah are working on taking the derivative of

$$f(x) = \frac{2x}{3x + 4}.$$

Katie uses the quotient rule to get

$$f'(x) = \frac{(3x + 4)2 - 2x(3)}{(3x + 4)^2} = \frac{8}{(3x + 4)^2}.$$

Sarah converts it into a product and uses the product rule and the chain rule:

$$f(x) = 2x(3x + 4)^{-1}$$
$$f'(x) = 2x(-1)(3x + 4)^{-2}(3) + 2(3x + 4)^{-1}$$
$$= 2(3x + 4)^{-1} - 6x(3x + 4)^{-2}.$$

Explain the discrepancies between the two answers. Which procedure do you think is preferable?

52. Margy and Nate are working on taking the derivative of

$$f(x) = \frac{2}{(3x + 1)^4}.$$

Margy uses the quotient rule and chain rule as follows:

$$f'(x) = \frac{(3x + 1)^4 \cdot 0 - 2 \cdot 4(3x + 1)^3 \cdot 3}{(3x + 1)^8}$$

$$= \frac{-24(3x + 1)^3}{(3x + 1)^8} = \frac{-24}{(3x + 1)^5}.$$

Nate rewrites the function and uses the power rule and chain rule as follows:

$$f(x) = 2(3x + 1)^{-4}$$

$$f'(x) = (-4)2(3x + 1)^{-5} \cdot 3 = \frac{-24}{(3x + 1)^5}.$$

Compare the two procedures. Which procedure do you think is preferable?

APPLICATIONS

Business and Economics

53. Demand Suppose the demand for a certain brand of vacuum cleaner is given by

$$D(p) = \frac{-p^2}{100} + 500,$$

where p is the price in dollars. If the price, in terms of the cost c, is expressed as

$$p(c) = 2c - 10,$$

find the demand in terms of the cost.

54. Revenue Assume that the total revenue (in dollars) from the sale of x television sets is given by

$$R(x) = 24(x^2 + x)^{2/3}.$$

Find the marginal revenue when the following numbers of sets are sold.

a. 100 **b.** 200 **c.** 300

d. Find the average revenue from the sale of x sets.

e. Find the marginal average revenue.

f. Write a paragraph covering the following questions. How does the revenue change over time? What does the marginal revenue function tell you about the revenue function? What does the average revenue function tell you about the revenue function?

55. Interest A sum of \$1500 is deposited in an account with an interest rate of r percent per year, compounded daily. At the end of 5 years, the balance in the account is given by

$$A = 1500\left(1 + \frac{r}{36{,}500}\right)^{1825}.$$

Find the rate of change of A with respect to r for the followin interest rates.

a. 6% **b.** 8% **c.** 9%

56. Demand Suppose a demand function is given by

$$q = D(p) = 30\left(5 - \frac{p}{\sqrt{p^2 + 1}}\right),$$

where q is the demand for a product and p is the price per item in dollars. Find the rate of change in the demand for the product per unit change in price (i.e., find dq/dp).

57. Depreciation A certain truck depreciates according to the formula

$$V = \frac{60{,}000}{1 + 0.3t + 0.1t^2},$$

where V is the value of the truck (in dollars), t is time measured in years, and $t = 0$ represents the time of purchase (in years). Find the rate at which the value of the truck is changing at the following times.

a. 2 years **b.** 4 years

58. Cost Suppose the cost in dollars of manufacturing q items is given by

$$C = 2000q + 3500,$$

and the demand equation is given by

$$q = \sqrt{15{,}000 - 1.5p}.$$

In terms of the demand q,

a. find an expression for the revenue R;

b. find an expression for the profit P;

c. find an expression for the marginal profit.

d. Determine the value of the marginal profit when the price is \$5000.

Life Sciences

59. Fish Population Suppose the population P of a certain species of fish depends on the number x (in hundreds) of a smaller fish that serves as its food supply, so that

$$P(x) = 2x^2 + 1.$$

Suppose, also, that the number of the smaller species of fish depends on the amount a (in appropriate units) of its food supply, a kind of plankton. Specifically,

$$x = f(a) = 3a + 2.$$

A biologist wants to find the relationship between the population P of the large fish and the amount a of plankton available, that is, $P[f(a)]$. What is the relationship?

60. Oil Pollution An oil well off the Gulf Coast is leaking, with the leak spreading oil over the surface as a circle. At any time t (in minutes) after the beginning of the leak, the radius of the circular oil slick on the surface is $r(t) = t^2$ feet. Let $A(r) = \pi r^2$ represent the area of a circle of radius r.

a. Find and interpret $A[r(t)]$.

b. Find and interpret $D_t A[r(t)]$ when $t = 100$.

APPLY IT

(b) Find the rate of change of sales after 4 years.

SOLUTION The derivative of this sales function, which gives the rate of change of sales, was found in Example 3. Using that derivative,

$$S'(4) = \frac{3{,}000{,}000e^{-0.3(4)}}{[1 + 100e^{-0.3(4)}]^2} = \frac{3{,}000{,}000e^{-1.2}}{(1 + 100e^{-1.2})^2}.$$

t	$S(t)$
0	990
5	4300
10	17,000
15	47,000
20	80,000
30	99,000

Using a calculator, $e^{-1.2} \approx 0.3012$, and

$$S'(4) \approx \frac{3{,}000{,}000(0.3012)}{[1 + 100(0.3012)]^2}$$

$$\approx \frac{903{,}600}{(1 + 30.12)^2}$$

$$\approx \frac{903{,}600}{968.5} \approx 933.$$

The rate of change of sales after 4 years is about 933 units per year. The positive number indicates that sales are increasing at this time.

The graph of the function in Example 5 is shown in Figure 11.

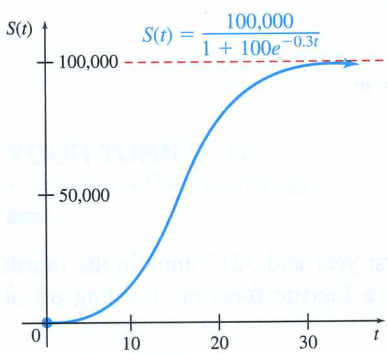


FIGURE 11

12.4 EXERCISES

Find derivatives of the functions defined as follows.

1. $y = e^{4x}$

2. $y = e^{-2x}$

3. $y = -8e^{3x}$

4. $y = 1.2e^{5x}$

5. $y = -16e^{2x+1}$

6. $y = -4e^{-0.3x}$

7. $y = e^{x^2}$

8. $y = e^{-x^2}$

9. $y = 3e^{2x^2}$

10. $y = -5e^{4x^3}$

11. $y = 4e^{2x^2-4}$

12. $y = -3e^{3x^2+5}$

13. $y = xe^x$

14. $y = x^2e^{-2x}$

15. $y = (x+3)^2e^{4x}$

16. $y = (3x^3 - 4x)e^{-5x}$

17. $y = \dfrac{x^2}{e^x}$

18. $y = \dfrac{e^x}{2x+1}$

19. $y = \dfrac{e^x + e^{-x}}{x}$

20. $y = \dfrac{e^x - e^{-x}}{x}$

21. $p = \dfrac{10{,}000}{9 + 4e^{-0.2t}}$

22. $p = \dfrac{500}{12 + 5e^{-0.5t}}$

23. $f(z) = (2z + e^{-z^2})^2$

24. $f(t) = (e^{t^2} + 5t)^3$

25. $y = 7^{3x+1}$

26. $y = 4^{-5x+2}$

27. $y = 3 \cdot 4^{x^2+2}$

28. $y = -10^{3x^2-4}$

29. $s = 2 \cdot 3^{\sqrt{t}}$

30. $s = 5 \cdot 2^{\sqrt{t-2}}$

31. $y = \dfrac{te^t + 2}{e^{2t} + 1}$

32. $y = \dfrac{t^2e^{2t}}{t + e^{3t}}$

33. $f(x) = e^{x\sqrt{3x+2}}$

34. $f(x) = e^{x^2/(x^3+2)}$

35. Prove that if $y = y_0e^{kt}$, where y_0 and k are constants, then $dy/dt = ky$. (This says that for exponential growth and decay, the rate of change of the population is proportional to the size of the population, and the constant of proportionality is the growth or decay constant.)

36. Use a graphing calculator to sketch the graph of $y = [f(x+h) - f(x)]/h$ using $f(x) = e^x$ and $h = 0.0001$. Compare it with the graph of $y = e^x$ and discuss what you observe.

37. Use graphical differentiation to verify that $\dfrac{d}{dx}(e^x) = e^x$.

APPLICATIONS

Business and Economics

38. **Sales** The sales of a new personal computer (in thousands) are given by

$$S(t) = 100 - 90e^{-0.3t},$$

where t represents time in years. Find the rate of change of sales at each time.

a. After 1 year **b.** After 5 years

c. What is happening to the rate of change of sales as time goes on?

d. Does the rate of change of sales ever equal zero?

39. **Cost** The cost in dollars to produce x DVDs can be approximated by

$$C(x) = \sqrt{900 - 800 \cdot 1.1^{-x}}.$$

Find the marginal cost when the following quantities are made.

a. 0 **b.** 20

c. What happens to the marginal cost as the number produced becomes larger and larger?

40. **Product Awareness** After the introduction of a new product for tanning without sun exposure, the percent of the public that is aware of the product is approximated by

$$A(t) = 10t^2 2^{-t},$$

where t is the time in months. Find the rate of change of the percent of the public that is aware of the product after the following numbers of months.

a. 2 **b.** 4

c. Notice that the answer to part a is positive and the answer to part b is negative. What does this tell you about how public awareness of the product has changed?

41. **Product Durability** Using data in a car magazine, we constructed the mathematical model

$$y = 100e^{-0.03045t}$$

for the percent of cars of a certain type still on the road after t years. Find the percent of cars on the road after the following numbers of years.

a. 0 **b.** 2 **c.** 4 **d.** 6

Find the rate of change of the percent of cars still on the road after the following numbers of years.

e. 0 **f.** 2

g. Interpret your answers to parts e and f.

42. **Investment** The value of a particular investment changes over time according to the function

$$S(t) = 5000e^{0.1(e^{0.25t})},$$

where $S(t)$ is the value after t years. Calculate the rate at which the value of the investment is changing after 8 years. (Choose one of the following.) *Source: Society of Actuaries.*

a. 618 **b.** 1934 **c.** 2011 **d.** 7735 **e.** 10,468

43. **Internet Users** The growth in the number (in millions) of Internet users in the United States between 1990 and 2015 can be approximated by a logistic function with $k = 0.0018$, where t is the number of years since 1990. In 1990 (when $t = 0$), the number of users was about 2 million, and the number is expected to level out around 250 million. *Source: World Bank.*

a. Find the growth function $G(t)$ for the number of Internet users in the United States.

Estimate the number of Internet users in the United States and the rate of growth for the following years.

b. 1995

c. 2000

d. 2010

e. What happens to the rate of growth over time?

Life Sciences

44. **Population Growth** In Section 10.4, Exercise 47, the growth in world population (in millions) was approximated by the exponential function

$$A(t) = 3100e^{0.0166t},$$

where t is the number of years since 1960. Find the instantaneous rate of change in the world population at the following times. *Source: United Nations.*

a. 2010 **b.** 2015

45. **Minority Population** In Section 10.4, Exercise 49, we saw that the projected Hispanic population in the United States (in millions) can be approximated by the function

$$h(t) = 37.79(1.021)^t$$

where $t = 0$ corresponds to 2000 and $0 \le t \le 50$. *Source: U.S. Census Bureau.*

a. Estimate the Hispanic population in the United States for the year 2015.

b. What is the instantaneous rate of change of the Hispanic population in the United States when $t = 15$? Interpret your answer.

46. **Insect Growth** The growth of a population of rare South American beetles is given by the logistic function with $k = 0.00001$ and t in months. Assume that there are 200 beetles initially and that the maximum population size is 10,000.

a. Find the growth function $G(t)$ for these beetles.

Find the population and rate of growth of the population after the following times.

b. 6 months **c.** 3 years **d.** 7 years

e. What happens to the rate of growth over time?

47. Clam Population The population of a bed of clams in the Great South Bay off Long Island is described by the logistic function with $k = 0.0001$ and t in years. Assume that there are 400 clams initially and that the maximum population size is 5200.

a. Find the growth function $G(t)$ for the clams.

Find the population and rate of growth of the population after the following times.

b. 1 year **c.** 4 years **d.** 10 years

e. What happens to the rate of growth over time?

48. Pollution Concentration The concentration of pollutants (in grams per liter) in the east fork of the Big Weasel River is approximated by

$$P(x) = 0.04e^{-4x},$$

where x is the number of miles downstream from a paper mill that the measurement is taken. Find the following values.

a. The concentration of pollutants 0.5 mile downstream

b. The concentration of pollutants 1 mile downstream

c. The concentration of pollutants 2 miles downstream

Find the rate of change of concentration with respect to distance for the following distances.

d. 0.5 mile **e.** 1 mile **f.** 2 miles

49. Breast Cancer It has been observed that the following formula accurately models the relationship between the size of a breast tumor and the amount of time that it has been growing.

$$V(t) = 1100[1023e^{-0.02415t} + 1]^{-4},$$

where t is in months and $V(t)$ is measured in cubic centimeters. *Source: Cancer.*

a. Find the tumor volume at 240 months.

b. Assuming that the shape of a tumor is spherical, find the radius of the tumor from part a. (*Hint:* The volume of a sphere is given by the formula $V = (4/3)\pi r^3$.)

c. If a tumor of size 0.5 cm^3 is detected, according to the formula, how long has it been growing? What does this imply?

d. Find $\lim_{t\to\infty} V(t)$ and interpret this value. Explain whether this makes sense.

e. Calculate the rate of change of tumor volume at 240 months and interpret.

50. Mortality The percentage of people of any particular age group that will die in a given year may be approximated by the formula

$$P(t) = 0.00239e^{0.0957t},$$

where t is the age of the person in years. *Source: U.S. Vital Statistics.*

a. Find $P(25)$, $P(50)$, and $P(75)$.

b. Find $P'(25)$, $P'(50)$, and $P'(75)$.

c. Interpret your answers for parts a and b. Are there any limitations of this formula?

51. Medical Literature It has been observed that there has bee[n an] increase in the proportion of medical research papers tha[t use] the word "novel" in the title or abstract, and that this propo[rtion] can be accurately modeled by the function

$$p(x) = 0.001131e^{0.1268x},$$

where x is the number of years since 1970. *Source: Nature.*

a. Find $p(40)$.

b. If this phenomenon continues, estimate the year in wh[ich] every medical article will contain the word "novel" in [its] title or abstract.

c. Estimate the rate of increase in the proportion of medi[cal] papers using this word in the year 2010.

d. Explain some factors that may be contributing to research[ers] using this word.

52. Arctic Foxes The age/weight relationship of female Arc[tic] foxes caught in Svalbard, Norway, can be estimated by t[he] function

$$M(t) = 3102e^{-e^{-0.022(t-56)}},$$

where t is the age of the fox in days and $M(t)$ is the weight [of] the fox in grams. *Source: Journal of Mammalogy.*

a. Estimate the weight of a female fox that is 200 days old.

b. Use $M(t)$ to estimate the largest size that a female fox ca[n] attain. (*Hint:* Find $\lim_{t\to\infty} M(t)$.)

c. Estimate the age of a female fox when it has reached 80% [of] its maximum weight.

d. Estimate the rate of change in weight of an Arctic fox that i[s] 200 days old. (*Hint:* Recall that $D_t[e^{f(t)}] = f'(t)e^{f(t)}$.)

e. Use a graphing calculator to graph $M(t)$ and then describ[e] the growth pattern.

f. Use the table function on a graphing calculator or a spread[-] sheet to develop a chart that shows the estimated weight an[d] growth rate of female foxes for days 50, 100, 150, 200, 25[0,] and 300.

53. Beef Cattle Researchers have compared two models that ar[e] used to predict the weight of beef cattle of various ages,

$$W_1(t) = 509.7(1 - 0.941e^{-0.00181t})$$

and

$$W_2(t) = 498.4(1 - 0.889e^{-0.00219t})^{1.25},$$

where $W_1(t)$ and $W_2(t)$ represent the weight (in kilograms) of a t-day-old beef cow. *Source: Journal of Animal Science.*

a. What is the maximum weight predicted by each function for the average beef cow? Is this difference significant?

b. According to each function, find the age that the average beef cow reaches 90% of its maximum weight.

c. Find $W_1'(750)$ and $W_2'(750)$. Compare your results.

d. Graph the two functions on $[0, 2500]$ by $[0, 525]$ and comment on the differences in growth patterns for each of these functions.

e. Graph the derivative of these two functions on $[0, 2500]$ by $[0, 1]$ and comment on any differences you notice between these functions.

54. Cholesterol ... heart disease ... may be appr...

$R($

where R is t... per 1000 p... a person's ... 15 mg/dL ... heart disea...

Social Sci...

55. Survival ... demonstr... modeled ... copies c... found to ... its publ... tion of ... limiting ... Bede's ... *Source*

a. Fin... *De*

Find ... after ...

b. 1

e. W

56. Habi... C. L... time ... $H($

wh... of

a.

d.

57. O...

EXAMPLE 4 **Automobile Resale Value**

Based on projections from the *Kelly Blue Book*, the resale value of a 2010 Toyota Coro 4-door sedan can be approximated by the following function

$$f(x) = 15{,}450 - 13{,}915 \log(t + 1),$$

where t is the number of years since 2010. Find and interpret $f(4)$ and $f'(4)$. *Sour* *Kelly Blue Book.*

APPLY IT

SOLUTION Recognizing this function as a common (base 10) logarithm, we have

$$f(4) = 15{,}450 - 13{,}915 \log(4 + 1) \approx 5500.08.$$

The average resale value of a 2010 Toyota Corolla in 2014 would be approximatel $5500.08.

The derivative of $f(t)$ is

$$f'(t) = \frac{-13{,}915}{(\ln 10)(t + 1)},$$

so $f'(4) \approx -1208.64$. In 2014, the average resale value of a 2010 Toyota Corolla i decreasing by $1208.64 per year.

12.5 EXERCISES

Find the derivative of each function.

1. $y = \ln(8x)$
2. $y = \ln(-4x)$
3. $y = \ln(8 - 3x)$
4. $y = \ln(1 + x^3)$
5. $y = \ln|4x^2 - 9x|$
6. $y = \ln|-8x^3 + 2x|$
7. $y = \ln\sqrt{x + 5}$
8. $y = \ln\sqrt{2x + 1}$
9. $y = \ln(x^4 + 5x^2)^{3/2}$
10. $y = \ln(5x^3 - 2x)^{3/2}$
11. $y = -5x\ln(3x + 2)$
12. $y = (3x + 7)\ln(2x - 1)$
13. $s = t^2\ln|t|$
14. $y = x\ln|2 - x^2|$
15. $y = \dfrac{2\ln(x + 3)}{x^2}$
16. $v = \dfrac{\ln u}{u^3}$
17. $y = \dfrac{\ln x}{4x + 7}$
18. $y = \dfrac{-2\ln x}{3x - 1}$
19. $y = \dfrac{3x^2}{\ln x}$
20. $y = \dfrac{x^3 - 1}{2\ln x}$
21. $y = (\ln|x + 1|)^4$
22. $y = \sqrt{\ln|x - 3|}$
23. $y = \ln|\ln x|$
24. $y = (\ln 4)(\ln|3x|)$
25. $y = e^{x^2}\ln x$
26. $y = e^{2x-1}\ln(2x - 1)$
27. $y = \dfrac{e^x}{\ln x}$
28. $p(y) = \dfrac{\ln y}{e^y}$
29. $g(z) = (e^{2z} + \ln z)^3$
30. $s(t) = \sqrt{e^{-t} + \ln 2t}$
31. $y = \log(6x)$
32. $y = \log(4x - 3)$
33. $y = \log|1 - x|$
34. $y = \log|3x|$
35. $y = \log_5\sqrt{5x + 2}$
36. $y = \log_7\sqrt{4x - 3}$
37. $y = \log_3(x^2 + 2x)^{3/2}$
38. $y = \log_2(2x^2 - x)^{5/2}$
39. $w = \log_8(2^p - 1)$
40. $z = 10^y \log y$
41. $f(x) = e^{\sqrt{x}}\ln(\sqrt{x} + 5)$
42. $f(x) = \ln(xe^{\sqrt{x}} + 2)$
43. $f(t) = \dfrac{\ln(t^2 + 1) + t}{\ln(t^2 + 1) + 1}$
44. $f(t) = \dfrac{2t^{3/2}}{\ln(2t^{3/2} + 1)}$
45. Why do we use the absolute value of x or of $g(x)$ in the derivative formulas for the natural logarithm?
46. Prove $\dfrac{d}{dx}\ln|ax| = \dfrac{d}{dx}\ln|x|$ for any constant a.
47. A friend concludes that because $y = \ln 6x$ and $y = \ln x$ have the same derivative, namely $dy/dx = 1/x$, these two functions must be the same. Explain why this is incorrect.
48. Use a graphing calculator to sketch the graph of $y = [f(x + h) - f(x)]/h$ using $f(x) = \ln|x|$ and $h = 0.0001$. Compare it with the graph of $y = 1/x$ and discuss what you observe.
49. Using the fact that

$$\ln[u(x)v(x)] = \ln u(x) + \ln v(x),$$

use the chain rule and the formula for the derivative of $\ln x$ to derive the product rule. In other words, find $[u(x)v(x)]'$ without assuming the product rule.

50. Using the fact that

$$\ln \frac{u(x)}{v(x)} = \ln u(x) - \ln v(x),$$

use the chain rule and the formula for the derivative of $\ln x$ to derive the quotient rule. In other words, find $[u(x)/v(x)]'$ without assuming the quotient rule.

51. Use graphical differentiation to verify that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

52. Use the fact that $d \ln x/dx = 1/x$, as well as the change-of-base theorem for logarithms, to prove that

$$\frac{d \log_a x}{dx} = \frac{1}{x \ln a}.$$

53. Let

$$h(x) = u(x)^{v(x)}.$$

a. Using the fact that

$$\ln [u(x)^{v(x)}] = v(x) \ln u(x),$$

use the chain rule, the product rule, and the formula for the derivative of $\ln x$ to show that

$$\frac{d}{dx} \ln h(x) = \frac{v(x)u'(x)}{u(x)} + (\ln u(x))\, v'(x).$$

b. Use the result from part a and the fact that

$$\frac{d}{dx} \ln h(x) = \frac{h'(x)}{h(x)}$$

to show that

$$\frac{d}{dx} h(x) = u(x)^{v(x)} \left[\frac{v(x)u'(x)}{u(x)} + (\ln u(x))\, v'(x) \right].$$

The idea of taking the logarithm of a function before differentiating is known as logarithmic differentiation.

Use the ideas from Exercise 53 to find the derivative of each of the following functions.

54. $h(x) = x^x$

55. $h(x) = (x^2 + 1)^{5x}$

APPLICATIONS

Business and Economics

56. **Profit** If the total revenue received from the sale of x items is given by

$$R(x) = 30 \ln(2x + 1),$$

while the total cost to produce x items is $C(x) = x/2$, find the following.

a. The marginal revenue

b. The profit function $P(x)$

c. The marginal profit when $x = 60$

d. Interpret the results of part c.

57. **Revenue** Suppose the demand function for q units of a certain item is

$$p = D(q) = 100 + \frac{50}{\ln q}, \quad q > 1,$$

where p is in dollars.

a. Find the marginal revenue.

b. Approximate the revenue from one more unit when 8 units are sold.

c. How might a manager use the information from part b?

58. **Profit** If the cost function in dollars for q units of the item in Exercise 57 is $C(q) = 100q + 100$, find the following.

a. The marginal cost

b. The profit function $P(q)$

c. The approximate profit from one more unit when 8 units are sold

d. How might a manager use the information from part c?

59. **Marginal Average Cost** Suppose the cost in dollars to make x oboe reeds is given by

$$C(x) = 5 \log_2 x + 10.$$

Find the marginal average cost when the following numbers of reeds are sold.

a. 10 b. 20

Life Sciences

60. **Body Surface Area** There is a mathematical relationship between an infant's weight and total body surface area (BSA), given by

$$A(w) = 4.688w^{0.8168 - 0.0154 \log_{10} w},$$

where w is the weight (in grams) and $A(w)$ is the BSA in square centimeters. *Source: British Journal of Cancer.*

a. Find the BSA for an infant who weighs 4000 g.

b. Find $A'(4000)$ and interpret your answer.

c. Use a graphing calculator to graph $A(w)$ on $[2000, 10{,}000]$ by $[0, 6000]$.

61. **Bologna Sausage** Scientists have developed a model to predict the growth of bacteria in bologna sausage at 32°C. The number of bacteria is given by

$$\ln\left(\frac{N(t)}{N_0}\right) = 9.8901e^{-e^{2.54197 - 0.2167t}},$$

where N_0 is the number of bacteria present at the beginning of the experiment and $N(t)$ is the number of bacteria present at time t (in hours). *Source: Applied and Environmental Microbiology.*

a. Use the properties of logarithms to find an expression for $N(t)$. Assume that $N_0 = 1000$.

b. Use a graphing calculator to estimate the derivative of $N(t)$ when $t = 20$ and interpret.

c. Let $S(t) = \ln(N(t)/N_0)$. Graph $S(t)$ on $[0, 35]$ by $[0, 12]$.

d. Graph $N(t)$ on $[0, 35]$ by $[0, 20{,}000{,}000]$ and compare the graphs from parts c and d.

e. Find $\lim_{t\to\infty} S(t)$ and then use this limit to find $\lim_{t\to\infty} N(t)$.

62. Pronghorn Fawns The field metabolic rate (FMR), or the total energy expenditure per day in excess of growth, can be calculated for pronghorn fawns using Nagy's formula,

$$F(x) = 0.774 + 0.727 \log x,$$

where x is the mass (in grams) of the fawn and $F(x)$ is the energy expenditure (in kJ/day). *Source: Animal Behavior.*

a. Determine the total energy expenditure per day in excess of growth for a pronghorn fawn that weighs 25,000 g.

b. Find $F'(25{,}000)$ and interpret the result.

c. Graph the function on $[5000, 30{,}000]$ by $[3, 5]$.

63. Fruit Flies A study of the relation between the rate of reproduction in *Drosophila* (fruit flies) bred in bottles and the density of the mated population found that the number of imagoes (sexually mature adults) per mated female per day (y) can be approximated by

$$\log y = 1.54 - 0.008x - 0.658 \log x,$$

where x is the mean density of the mated population (measured as flies per bottle) over a 16-day period. *Source: Elements of Mathematical Biology.*

a. Show that the above equation is equivalent to

$$y = 34.7(1.0186)^{-x}x^{-0.658}.$$

b. Using your answer from part a, find the number of imagoes per mated female per day when the density is

i. 20 flies per bottle;

ii. 40 flies per bottle.

c. Using your answer from part a, find the rate of change in the number of imagoes per mated female per day with respect to the density when the density is

i. 20 flies per bottle;

ii. 40 flies per bottle.

64. Insect Mating Consider an experiment in which equal numbers of male and female insects of a certain species are permitted to intermingle. Assume that

$$M(t) = (0.1t + 1)\ln\sqrt{t}$$

represents the number of matings observed among the insects in an hour, where t is the temperature in degrees Celsius. (*Note:* The formula is an approximation at best and holds only for specific temperature intervals.)

a. Find the number of matings when the temperature is 15°C.

b. Find the number of matings when the temperature is 25°C.

c. Find the rate of change of the number of matings when the temperature is 15°C.

65. Population Growth Suppose that the population of a certain collection of rare Brazilian ants is given by

$$P(t) = (t + 100)\ln(t + 2),$$

where t represents the time in days. Find the rates of change of the population on the second day and on the eighth day.

Social Sciences

66. Poverty The passage of the Social Security Amendments of 1965 resulted in the creation of the Medicare and Medicaid programs. Since then, the percent of persons 65 years and over with family income below the poverty level has declined. The percent can be approximated by the following function:

$$P(t) = 30.60 - 5.79 \ln t,$$

where t is the number of years since 1965. Find the percent of persons 65 years and over with family income below the poverty level and the rate of change in the following years. *Source: U.S. Census.*

a. 1970 **b.** 1990 **c.** 2010

d. What happens to the rate of change over time?

Physical Sciences

67. Richter Scale The Richter scale provides a measure of the magnitude of an earthquake. In fact, the largest Richter number M ever recorded for an earthquake was 8.9 from the 1933 earthquake in Japan. The following formula shows a relationship between the amount of energy released and the Richter number.

$$M = \frac{2}{3}\log\frac{E}{0.007},$$

where E is measured in kilowatt-hours. *Source: Mathematics Teacher.*

a. For the 1933 earthquake in Japan, what value of E gives a Richter number $M = 8.9$?

b. If the average household uses 247 kWh per month, how many months would the energy released by an earthquake of this magnitude power 10 million households?

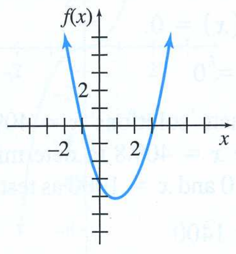
c. Find the rate of change of the Richter number M with respect to energy when $E = 70{,}000$ kWh.

d. What happens to dM/dE as E increases?

13.1 EXERCISES

Find the open intervals where the functions graphed as follows are (a) increasing, or (b) decreasing.

1.



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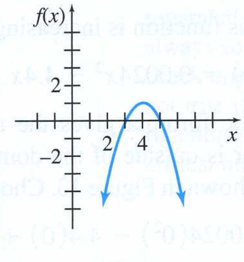

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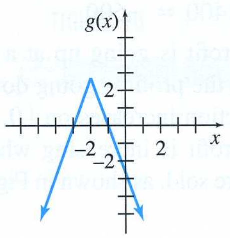

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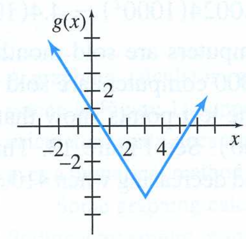

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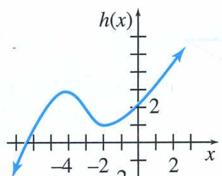

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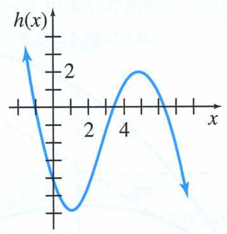

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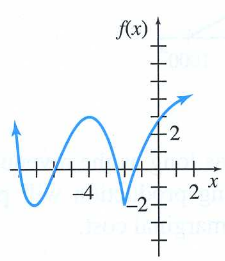

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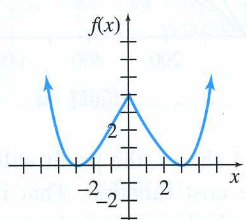


For each of the exercises listed below, suppose that the function that is graphed is not $f(x)$, but $f'(x)$. Find the open intervals where $f(x)$ is (a) increasing or (b) decreasing.

9. Exercise 1
10. Exercise 2
11. Exercise 7
12. Exercise 8

For each function, find (a) the critical numbers; (b) the open intervals where the function is increasing; and (c) the open intervals where it is decreasing.

13. $y = 2.3 + 3.4x - 1.2x^2$

14. $y = 1.1 - 0.3x - 0.3x^2$

15. $f(x) = \frac{2}{3}x^3 - x^2 - 24x - 4$

16. $f(x) = \frac{2}{3}x^3 - x^2 - 4x + 2$

17. $f(x) = 4x^3 - 15x^2 - 72x + 5$

18. $f(x) = 4x^3 - 9x^2 - 30x + 6$

19. $f(x) = x^4 + 4x^3 + 4x^2 + 1$

20. $f(x) = 3x^4 + 8x^3 - 18x^2 + 5$

21. $y = -3x + 6$

22. $y = 6x - 9$

23. $f(x) = \frac{x+2}{x+1}$

24. $f(x) = \frac{x+3}{x-4}$

25. $y = \sqrt{x^2 + 1}$

26. $y = x\sqrt{9 - x^2}$

27. $f(x) = x^{2/3}$

28. $f(x) = (x+1)^{4/5}$

29. $y = x - 4\ln(3x - 9)$

30. $f(x) = \ln\frac{5x^2 + 4}{x^2 + 1}$

31. $f(x) = xe^{-3x}$

32. $f(x) = xe^{x^2 - 3x}$

33. $f(x) = x^2 2^{-x}$

34. $f(x) = x2^{-x^2}$

35. $y = x^{2/3} - x^{5/3}$

36. $y = x^{1/3} + x^{4/3}$

37. A friend looks at the graph of $y = x^2$ and observes that if you start at the origin, the graph increases whether you go to the right or the left, so the graph is increasing everywhere. Explain why this reasoning is incorrect.

38. Use the techniques of this chapter to find the vertex and intervals where f is increasing and decreasing, given

$$f(x) = ax^2 + bx + c,$$

where we assume $a > 0$. Verify that this agrees with what we found in Chapter 10.

39. Repeat Exercise 38 under the assumption $a < 0$.

40. Where is the function defined by $f(x) = e^x$ increasing? Decreasing? Where is the tangent line horizontal?

41. Repeat Exercise 40 with the function defined by $f(x) = \ln x$.

42. a. For the function in Exercise 15, find the average of the critical numbers.

b. For the function in Exercise 15, use a graphing calculator to find the roots of the function, and then find the average of those roots.

c. Compare your answers to parts a and b. What do you notice?

d. Repeat part a for the function in Exercise 17.

e. Repeat part b for the function in Exercise 17.

f. Compare your answers to parts d and e. What do you notice?

It can be shown that the average of the roots of a polynomial (including the complex roots, if there are any) and the critical numbers of a polynomial (including complex roots of $f'(x) = 0$, if there are any) are always equal. *Source: The Mathematics Teacher.*

For each of the following functions, use a graphing calculator to find the open intervals where $f(x)$ is (a) increasing, or (b) decreasing.

43. $f(x) = e^{0.001x} - \ln x$

44. $f(x) = \ln(x^2 + 1) - x^{0.3}$

APPLICATIONS

Business and Economics

45. Housing Starts A county realty group estimates that the number of housing starts per year over the next three years will be

$$H(r) = \frac{300}{1 + 0.03r^2},$$

where r is the mortgage rate (in percent).

a. Where is $H(r)$ increasing?

b. Where is $H(r)$ decreasing?

46. Cost Suppose the total cost $C(x)$ (in dollars) to manufacture a quantity x of weed killer (in hundreds of liters) is given by

$$C(x) = x^3 - 2x^2 + 8x + 50.$$

a. Where is $C(x)$ decreasing?

b. Where is $C(x)$ increasing?

47. Profit A manufacturer sells video games with the following cost and revenue functions (in dollars), where x is the number of games sold, for $0 \le x \le 3300$.

$$C(x) = 0.32x^2 - 0.00004x^3$$
$$R(x) = 0.848x^2 - 0.0002x^3$$

Determine the interval(s) on which the profit function is increasing.

48. Profit A manufacturer of CD players has determined that the profit $P(x)$ (in thousands of dollars) is related to the quantity x of CD players produced (in hundreds) per month by

$$P(x) = -(x - 4)e^x - 4, \qquad 0 < x \le 3.9.$$

a. At what production levels is the profit increasing?

b. At what levels is it decreasing?

49. Social Security Assets The projected year-end assets in the Social Security trust funds, in trillions of dollars, where t represents the number of years since 2000, can be approximated by

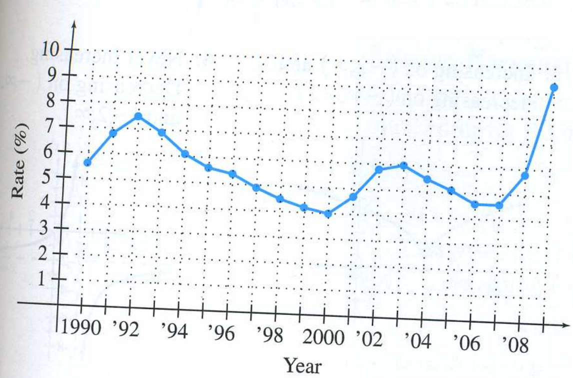
$$A(t) = 0.0000329t^3 - 0.00450t^2 + 0.0613t + 2.34,$$

where $0 \le t \le 50$. *Source: Social Security Administration.*

a. Where is $A(t)$ increasing?

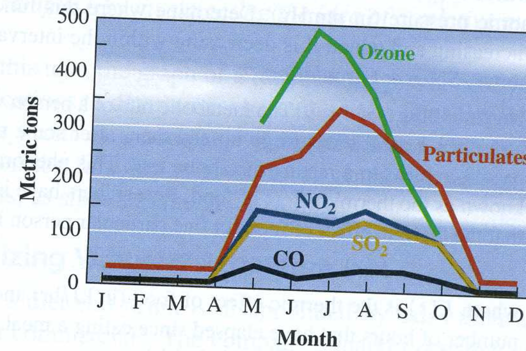
b. Where is $A(t)$ decreasing?

50. Unemployment The annual unemployment rates of the U.S. civilian noninstitutional population for 1990–2009 are shown in the graph. When is the function increasing? Decreasing? Constant? *Source: Bureau of Labor Statistics.*



Life Sciences

51. Air Pollution The graph shows the amount of air pollution removed by trees in the Chicago urban region for each month of the year. From the graph we see, for example, that the ozone level starting in May increases up to June, and then abruptly decreases. *Source: National Arbor Day Foundation.*



a. Are these curves the graphs of functions?

b. Look at the graph for particulates. Where is the function increasing? Decreasing? Constant?

c. On what intervals do all four lower graphs indicate that the corresponding functions are constant? Why do you think the functions are constant on those intervals?

52. Spread of Infection The number of people $P(t)$ (in hundreds) infected t days after an epidemic begins is approximated by

$$P(t) = \frac{10\ln(0.19t + 1)}{0.19t + 1}.$$

When will the number of people infected start to decline?

53. Alcohol Concentration In Exercise 55 in the section on Polynomial and Rational Functions, we gave the function defined by

$$A(x) = 0.003631x^3 - 0.03746x^2 + 0.1012x + 0.009$$

as the approximate blood alcohol concentration in a 170-lb woman x hours after drinking 2 oz of alcohol on an empty stomach, for x in the interval [0, 5]. *Source: Medicolegal Aspects of Alcohol Determination in Biological Specimens.*

a. On what time intervals is the alcohol concentration increasing?

b. On what intervals is it decreasing?

54. Drug Concentration The percent of concentration of a drug in the bloodstream x hours after the drug is administered is given by

$$K(x) = \frac{4x}{3x^2 + 27}.$$

a. On what time intervals is the concentration of the drug increasing?

b. On what intervals is it decreasing?

55. Drug Concentration Suppose a certain drug is administered to a patient, with the percent of concentration of the drug in the bloodstream t hours later given by

$$K(t) = \frac{5t}{t^2 + 1}.$$

a. On what time intervals is the concentration of the drug increasing?

b. On what intervals is it decreasing?

CAUTION Be careful to give the y-value of the point where an extremum occurs. Although we solve the equation $f'(x) = 0$ for x to find the extremum, the maximum or minimum value of the function is the corresponding y-value. Thus, in Example 4(a), we found that at $q = 45$, the maximum weekly revenue is \$2025 (not \$45).

The examples in this section involving the maximization of a quadratic function, such as the advertising example and the bicycle revenue example, could be solved by the methods described in Chapter 10 on Nonlinear Functions. But those involving more complicated functions, such as the bicycle profit example, are difficult to analyze without the tools of calculus.

Finding extrema for realistic problems requires an accurate mathematical model of the problem. In particular, it is important to be aware of restrictions on the values of the variables. For example, if $T(x)$ closely approximates the number of items that can be manufactured daily on a production line when x is the number of employees on the line, x must certainly be restricted to the positive integers or perhaps to a few common fractional values. (We can imagine half-time workers, but not 1/49-time workers.)

On the other hand, to apply the tools of calculus to obtain an extremum for some function, the function must be defined and be meaningful at every real number in some interval. Because of this, the answer obtained from a mathematical model might be a number that is not feasible in the actual problem.

Usually, the requirement that a continuous function be used, rather than one that can take on only certain selected values, is of theoretical interest only. In most cases, the methods of calculus give acceptable results as long as the assumptions of continuity and differentiability are not totally unreasonable. If they lead to the conclusion, say, that $80\sqrt{2}$ workers should be hired, it is usually only necessary to investigate acceptable values close to $80\sqrt{2}$. This was done in Example 4.

13.2 EXERCISES

Find the locations and values of all relative extrema for the functions with graphs as follows. Compare with Exercises 1–8 in the preceding section.

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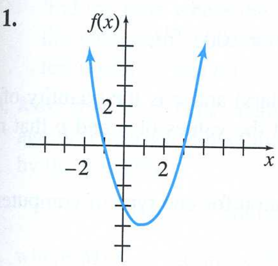

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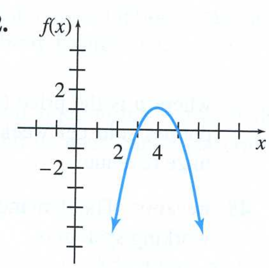

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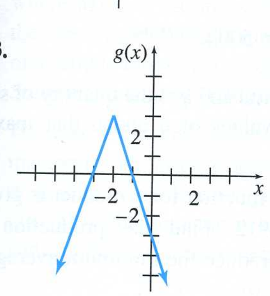

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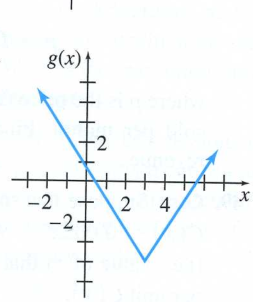

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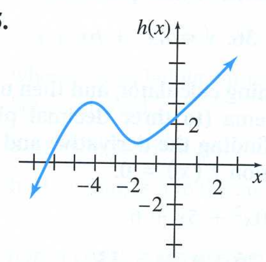

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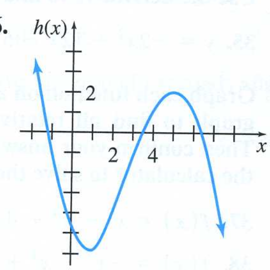

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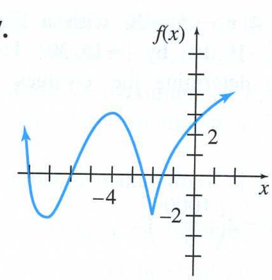

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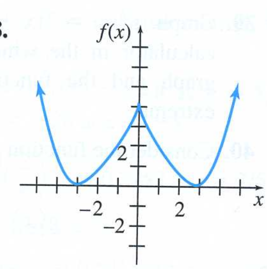

For each of the exercises listed below, suppose that the function that is graphed is not $f(x)$ but $f'(x)$. Find the locations of all relative extrema, and tell whether each extremum is a relative maximum or minimum.

9. Exercise 1

10. Exercise 2

11. Exercise 7

12. Exercise 8

Find the x-value of all points where the functions defined as follows have any relative extrema. Find the value(s) of any relative extrema.

13. $f(x) = x^2 - 10x + 33$

14. $f(x) = x^2 + 8x + 5$

15. $f(x) = x^3 + 6x^2 + 9x - 8$

16. $f(x) = x^3 + 3x^2 - 24x + 2$

17. $f(x) = -\frac{4}{3}x^3 - \frac{21}{2}x^2 - 5x + 8$

18. $f(x) = -\frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x - 4$

19. $f(x) = x^4 - 18x^2 - 4$

20. $f(x) = x^4 - 8x^2 + 9$

21. $f(x) = 3 - (8 + 3x)^{2/3}$

22. $f(x) = \frac{(5 - 9x)^{2/3}}{7} + 1$

23. $f(x) = 2x + 3x^{2/3}$

24. $f(x) = 3x^{5/3} - 15x^{2/3}$

25. $f(x) = x - \frac{1}{x}$

26. $f(x) = x^2 + \frac{1}{x}$

27. $f(x) = \frac{x^2 - 2x + 1}{x - 3}$

28. $f(x) = \frac{x^2 - 6x + 9}{x + 2}$

29. $f(x) = x^2e^x - 3$

30. $f(x) = 3xe^x + 2$

31. $f(x) = 2x + \ln x$

32. $f(x) = \frac{x^2}{\ln x}$

33. $f(x) = \frac{2^x}{x}$

34. $f(x) = x + 8^{-x}$

Use the derivative to find the vertex of each parabola.

35. $y = -2x^2 + 12x - 5$

36. $y = ax^2 + bx + c$

Graph each function on a graphing calculator, and then use the graph to find all relative extrema (to three decimal places). Then confirm your answer by finding the derivative and using the calculator to solve the equation $f'(x) = 0$.

37. $f(x) = x^5 - x^4 + 4x^3 - 30x^2 + 5x + 6$

38. $f(x) = -x^5 - x^4 + 2x^3 - 25x^2 + 9x + 12$

39. Graph $f(x) = 2|x + 1| + 4|x - 5| - 20$ with a graphing calculator in the window $[-10, 10]$ by $[-15, 30]$. Use the graph and the function to determine the x-values of all extrema.

40. Consider the function

$$g(x) = \frac{1}{x^{12}} - 2\left(\frac{1000}{x}\right)^6.$$

Source: Mathematics Teacher.

a. Using a graphing calculator, try to find any local minima, or tell why finding a local minimum is difficult for this function.

b. Find any local minima using the techniques of calculus.

c. Based on your results in parts a and b, describe circumstances under which relative extrema are easier to find using the techniques of calculus than using a graphing calculator.

APPLICATIONS

Business and Economics

Profit **In Exercises 41–44, find (a) the number, q, of units that produces maximum profit; (b) the price, p, per unit that produces maximum profit; and (c) the maximum profit, P.**

41. $C(q) = 80 + 18q;\quad p = 70 - 2q$

42. $C(q) = 25q + 5000;\quad p = 90 - 0.02q$

43. $C(q) = 100 + 20qe^{-0.01q};\quad p = 40e^{-0.01q}$

44. $C(q) = 21.047q + 3;\quad p = 50 - 5\ln(q + 10)$

45. Power On August 8, 2007, the power used in New York state (in thousands of megawatts) could be approximated by the function

$$P(t) = -0.005846t^3 + 0.1614t^2 - 0.4910t + 20.47,$$

where t is the number of hours since midnight, for $0 \le t \le 24$. Find any relative extrema for power usage, as well as when they occurred. *Source: Current Energy.*

46. Profit The total profit $P(x)$ (in thousands of dollars) from the sale of x units of a certain prescription drug is given by

$$P(x) = \ln(-x^3 + 3x^2 + 72x + 1)$$

for x in $[0, 10]$.

a. Find the number of units that should be sold in order to maximize the total profit.

b. What is the maximum profit?

47. Revenue The demand equation for telephones at one store is

$$p = D(q) = 200e^{-0.1q},$$

where p is the price (in dollars) and q is the quantity of telephones sold per week. Find the values of q and p that maximize revenue.

48. Revenue The demand equation for one type of computer networking system is

$$p = D(q) = 500qe^{-0.0016q^2},$$

where p is the price (in dollars) and q is the quantity of servers sold per month. Find the values of q and p that maximize revenue.

49. Cost Suppose that the cost function for a product is given by $C(x) = 0.002x^3 + 9x + 6912$. Find the production level (i.e., value of x) that will produce the minimum average cost per unit $\overline{C}(x)$.

Set $R''(x)$ equal to 0 and solve for x.

$$\frac{2}{25} - \frac{1}{2500}x = 0$$

$$-\frac{1}{2500}x = -\frac{2}{25}$$

$$x = \frac{5000}{25} = 200$$

Test a number in the interval $(0, 200)$ to see that $R''(x)$ is positive there. Then test a number in the interval $(200, 600)$ to find $R''(x)$ negative in that interval. Since the sign of $R''(x)$ changes from positive to negative at $x = 200$, the graph changes from concave upward to concave downward at that point, and there is a point of diminishing returns at the inflection point (200, $1066\frac{2}{3}$). Investments in advertising beyond \$200,000 return less and less for each dollar invested. Verify that $R'(200) = 8$. This means that when \$200,000 is invested, another \$1000 invested returns approximately \$8000 in additional revenue. Thus it may still be economically sound to invest in advertising beyond the point of diminishing returns.

13.3 EXERCISES

Find $f''(x)$ for each function. Then find $f''(0)$ and $f''(2)$.

1. $f(x) = 5x^3 - 7x^2 + 4x + 3$
2. $f(x) = 4x^3 + 5x^2 + 6x - 7$
3. $f(x) = 4x^4 - 3x^3 - 2x^2 + 6$
4. $f(x) = -x^4 + 7x^3 - \dfrac{x^2}{2}$
5. $f(x) = 3x^2 - 4x + 8$
6. $f(x) = 8x^2 + 6x + 5$
7. $f(x) = \dfrac{x^2}{1 + x}$
8. $f(x) = \dfrac{-x}{1 - x^2}$
9. $f(x) = \sqrt{x^2 + 4}$
10. $f(x) = \sqrt{2x^2 + 9}$
11. $f(x) = 32x^{3/4}$
12. $f(x) = -6x^{1/3}$
13. $f(x) = 5e^{-x^2}$
14. $f(x) = 0.5e^{x^2}$
15. $f(x) = \dfrac{\ln x}{4x}$
16. $f(x) = \ln x + \dfrac{1}{x}$

Find $f'''(x)$, the third derivative of f, and $f^{(4)}(x)$, the fourth derivative of f, for each function.

17. $f(x) = 7x^4 + 6x^3 + 5x^2 + 4x + 3$
18. $f(x) = -2x^4 + 7x^3 + 4x^2 + x$
19. $f(x) = 5x^5 - 3x^4 + 2x^3 + 7x^2 + 4$
20. $f(x) = 2x^5 + 3x^4 - 5x^3 + 9x - 2$
21. $f(x) = \dfrac{x - 1}{x + 2}$
22. $f(x) = \dfrac{x + 1}{x}$
23. $f(x) = \dfrac{3x}{x - 2}$
24. $f(x) = \dfrac{x}{2x + 1}$

25. Let $f(x) = \ln x$.

    a. Compute $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, and $f^{(5)}(x)$.

    b. Guess a formula for $f^{(n)}(x)$, where n is any positive integer.

26. For $f(x) = e^x$, find $f''(x)$ and $f'''(x)$. What is the nth derivative of f with respect to x?

In Exercises 27–48, find the open intervals where the functions are concave upward or concave downward. Find any inflection points.

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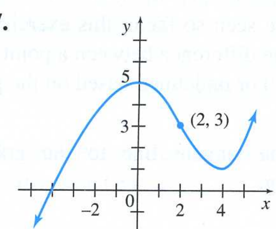

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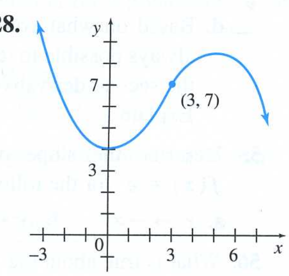

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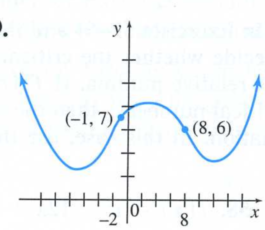
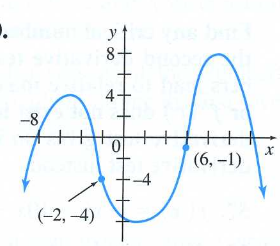


30.


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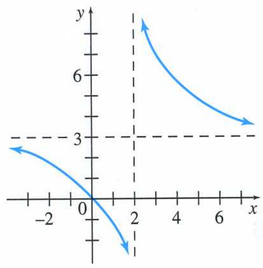

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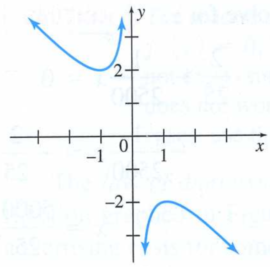


33. $f(x) = x^2 + 10x - 9$

34. $f(x) = 8 - 6x - x^2$

35. $f(x) = -2x^3 + 9x^2 + 168x - 3$

36. $f(x) = -x^3 - 12x^2 - 45x + 2$

37. $f(x) = \dfrac{3}{x - 5}$

38. $f(x) = \dfrac{-2}{x + 1}$

39. $f(x) = x(x + 5)^2$

40. $f(x) = -x(x - 3)^2$

41. $f(x) = 18x - 18e^{-x}$

42. $f(x) = 2e^{-x^2}$

43. $f(x) = x^{8/3} - 4x^{5/3}$

44. $f(x) = x^{7/3} + 56x^{4/3}$

45. $f(x) = \ln(x^2 + 1)$

46. $f(x) = x^2 + 8\ln|x + 1|$

47. $f(x) = x^2 \log|x|$

48. $f(x) = 5^{-x^2}$

For each of the exercises listed below, suppose that the function that is graphed is not $f(x)$, but $f'(x)$. Find the open intervals where the original function is concave upward or concave downward, and find the location of any inflection points.

49. Exercise 27

50. Exercise 28

51. Exercise 29

52. Exercise 30

53. Give an example of a function $f(x)$ such that $f'(0) = 0$ but $f''(0)$ does not exist. Is there a relative minimum or maximum or an inflection point at $x = 0$?

54. a. Graph the two functions $f(x) = x^{7/3}$ and $g(x) = x^{5/3}$ on the window $[-2, 2]$ by $[-2, 2]$.

b. Verify that both f and g have an inflection point at $(0, 0)$.

c. How is the value of $f''(0)$ different from $g''(0)$?

d. Based on what you have seen so far in this exercise, is it always possible to tell the difference between a point where the second derivative is 0 or undefined based on the graph? Explain.

55. Describe the slope of the tangent line to the graph of $f(x) = e^x$ for the following.

a. $x \to -\infty$

b. $x \to 0$

56. What is true about the slope of the tangent line to the graph of $f(x) = \ln x$ as $x \to \infty$? As $x \to 0$?

Find any critical numbers for f in Exercises 57–64 and then use the second derivative test to decide whether the critical numbers lead to relative maxima or relative minima. If $f''(c) = 0$ or $f''(c)$ does not exist for a critical number c, then the second derivative test gives no information. In this case, use the first derivative test instead.

57. $f(x) = -x^2 - 10x - 25$

58. $f(x) = x^2 - 12x + 36$

59. $f(x) = 3x^3 - 3x^2 + 1$

60. $f(x) = 2x^3 - 4x^2 + 2$

61. $f(x) = (x + 3)^4$

62. $f(x) = x^3$

63. $f(x) = x^{7/3} + x^{4/3}$

64. $f(x) = x^{8/3} + x^{5/3}$

Sometimes the derivative of a function is known, but not the function. We will see more of this later in the book. For each function f' defined in Exercises 65–68, find $f''(x)$, then use a graphing calculator to graph f' and f'' in the indicated window. Use the graph to do the following.

a. Give the (approximate) x-values where f has a maximum or minimum.

b. By considering the sign of $f'(x)$, give the (approximate) intervals where $f(x)$ is increasing and decreasing.

c. Give the (approximate) x-values of any inflection points.

d. By considering the sign of $f''(x)$, give the intervals where f is concave upward or concave downward.

65. $f'(x) = x^3 - 6x^2 + 7x + 4$; $[-5, 5]$ by $[-5, 15]$

66. $f'(x) = 10x^2(x - 1)(5x - 3)$; $[-1, 1.5]$ by $[-20, 20]$

67. $f'(x) = \dfrac{1 - x^2}{(x^2 + 1)^2}$; $[-3, 3]$ by $[-1.5, 1.5]$

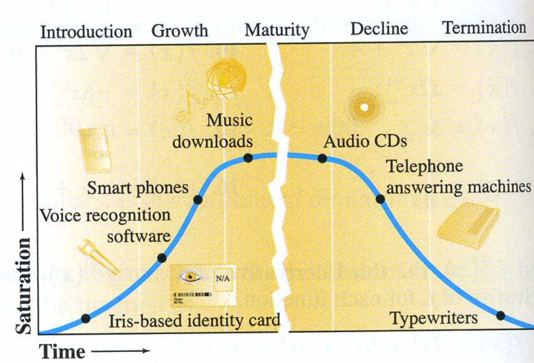
68. $f'(x) = x^2 + x\ln x$; $[0, 1]$ by $[-2, 2]$

69. Suppose a friend makes the following argument. A function f is increasing and concave downward. Therefore, f' is positive and decreasing, so it eventually becomes 0 and then negative, at which point f decreases. Show that your friend is wrong by giving an example of a function that is always increasing and concave downward.

APPLICATIONS

Business and Economics

70. **Product Life Cycle** The accompanying figure shows the *product life cycle* graph, with typical products marked on it. It illustrates the fact that a new product is often purchased at a faster and faster rate as people become familiar with it. In time, saturation is reached and the purchase rate stays constant until the product is made obsolete by newer products, after which it is purchased less and less. *Source: tutor2u.*



a. Which products on the left side of the graph are closest to the left-hand inflection point? What does the inflection point mean here?

b. Which product on the right side of the graph is closest to the right-hand inflection point? What does the inflection point mean here?

c. Discuss where portable Blu-ray players, iPads, and other new technologies should be placed on the graph.

To find any inflection points, we set $f''(x) = 0$.

$$f''(x) = \frac{x^3\left(-2 \cdot \frac{1}{x}\right) - (1 - 2\ln x)\cdot 3x^2}{(x^3)^2}$$

$$= \frac{-2x^2 - 3x^2 + 6x^2 \ln x}{x^6} = \frac{-5 + 6\ln x}{x^4}$$

$$\frac{-5 + 6\ln x}{x^4} = 0$$

$$-5 + 6\ln x = 0 \quad \text{Set the numerator equal to 0.}$$

$$6\ln x = 5 \quad \text{Add 5 to both sides.}$$

$$\ln x = 5/6 \quad \text{Divide both sides by 6.}$$

$$x = e^{5/6} \approx 2.3 \quad e^{\ln x} = x.$$

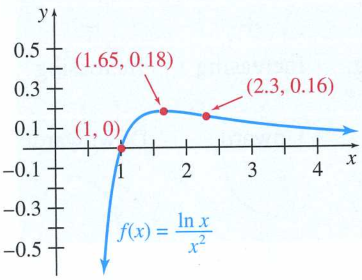


FIGURE 46

YOUR TURN 4 Graph $f(x) = (x + 2)e^{-x}$. (Recall $\lim_{x\to\infty} x^n e^{-x} = 0$)

There is an inflection point at $(2.3, f(2.3)) \approx (2.3, 0.16)$. Verify that $f''(1)$ is negative and $f''(3)$ is positive, so the graph is concave downward on $(1, 2.3)$ and upward on $(2.3, \infty)$. This information is summarized in the following table and could be used to sketch the graph. A graph of the function is shown in Figure 46. **TRY YOUR TURN 4**

Graph Summary

Interval	$(0, 1.65)$	$(1.65, 2.3)$	$(2.3, \infty)$
Sign of f'	+	−	−
Sign of f''	−	−	+
f Increasing or Decreasing	Increasing	Decreasing	Decreasing
Concavity of f	Downward	Downward	Upward
Shape of Graph			

As we saw earlier, a graphing calculator, when used with care, can be helpful in studying the behavior of functions. This section has illustrated that calculus is also a great help. The techniques of calculus show where the important points of a function, such as the relative extrema and the inflection points, are located. Furthermore, they tell how the function behaves between and beyond the points that are graphed, something a graphing calculator cannot always do.

13.4 EXERCISES

1. By sketching a graph of the function or by investigating values of the function near 0, find $\lim_{x\to 0} x \ln |x|$. (This result will be useful in Exercise 21.)

2. Describe how you would find the equation of the horizontal asymptote for the graph of
$$f(x) = \frac{3x^2 - 2x}{2x^2 + 5}.$$

Graph each function, considering the domain, critical points, symmetry, regions where the function is increasing or decreasing, inflection points, regions where the function is concave upward or concave downward, intercepts where possible, and asymptotes where applicable. (*Hint*: In Exercise 21, use the result of Exercise 1. In Exercises 25–27, recall from Exercise 66 in the section on Limits that $\lim_{x\to\infty} x^n e^{-x} = 0$.)

3. $f(x) = -2x^3 - 9x^2 + 108x - 10$

4. $f(x) = x^3 - \frac{15}{2}x^2 - 18x - 1$

5. $f(x) = -3x^3 + 6x^2 - 4x - 1$

6. $f(x) = x^3 - 6x^2 + 12x - 11$
7. $f(x) = x^4 - 24x^2 + 80$
8. $f(x) = -x^4 + 6x^2$
9. $f(x) = x^4 - 4x^3$
10. $f(x) = x^5 - 15x^3$
11. $f(x) = 2x + \dfrac{10}{x}$
12. $f(x) = 16x + \dfrac{1}{x^2}$
13. $f(x) = \dfrac{-x + 4}{x + 2}$
14. $f(x) = \dfrac{3x}{x - 2}$
15. $f(x) = \dfrac{1}{x^2 + 4x + 3}$
16. $f(x) = \dfrac{-8}{x^2 - 6x - 7}$
17. $f(x) = \dfrac{x}{x^2 + 1}$
18. $f(x) = \dfrac{1}{x^2 + 4}$
19. $f(x) = \dfrac{1}{x^2 - 9}$
20. $f(x) = \dfrac{-2x}{x^2 - 4}$
21. $f(x) = x \ln |x|$
22. $f(x) = x - \ln |x|$
23. $f(x) = \dfrac{\ln x}{x}$
24. $f(x) = \dfrac{\ln x^2}{x^2}$
25. $f(x) = xe^{-x}$
26. $f(x) = x^2 e^{-x}$
27. $f(x) = (x - 1)e^{-x}$
28. $f(x) = e^x + e^{-x}$
29. $f(x) = x^{2/3} - x^{5/3}$
30. $f(x) = x^{1/3} + x^{4/3}$

31. The default window on many calculators is $[-10, 10]$ by $[-10, 10]$. For the odd exercises between 3 and 15, tell which would give a poor representation in this window. (*Note:* Your answers may differ from ours, depending on what you consider "poor.")

32. Repeat Exercise 31 for the even exercises between 4 and 16.

33. Repeat Exercise 31 for the odd exercises between 17 and 29.

34. Repeat Exercise 31 for the even exercises between 18 and 30.

In Exercises 35–39, sketch the graph of a single function that has all of the properties listed.

35. **a.** Continuous and differentiable everywhere except at $x = 1$, where it has a vertical asymptote
b. $f'(x) < 0$ everywhere it is defined
c. A horizontal asymptote at $y = 2$
d. $f''(x) < 0$ on $(-\infty, 1)$ and $(2, 4)$
e. $f''(x) > 0$ on $(1, 2)$ and $(4, \infty)$

36. **a.** Continuous for all real numbers
b. $f'(x) < 0$ on $(-\infty, -6)$ and $(1, 3)$
c. $f'(x) > 0$ on $(-6, 1)$ and $(3, \infty)$
d. $f''(x) > 0$ on $(-\infty, -6)$ and $(3, \infty)$
e. $f''(x) < 0$ on $(-6, 3)$
f. A y-intercept at $(0, 2)$

37. **a.** Continuous and differentiable for all real numbers
b. $f'(x) > 0$ on $(-\infty, -3)$ and $(1, 4)$
c. $f'(x) < 0$ on $(-3, 1)$ and $(4, \infty)$
d. $f''(x) < 0$ on $(-\infty, -1)$ and $(2, \infty)$
e. $f''(x) > 0$ on $(-1, 2)$
f. $f'(-3) = f'(4) = 0$
g. $f''(x) = 0$ at $(-1, 3)$ and $(2, 4)$

38. **a.** Continuous for all real numbers
b. $f'(x) > 0$ on $(-\infty, -2)$ and $(0, 3)$
c. $f'(x) < 0$ on $(-2, 0)$ and $(3, \infty)$
d. $f''(x) < 0$ on $(-\infty, 0)$ and $(0, 5)$
e. $f''(x) > 0$ on $(5, \infty)$
f. $f'(-2) = f'(3) = 0$
g. $f'(0)$ doesn't exist
h. Differentiable everywhere except at $x = 0$
i. An inflection point at $(5, 1)$

39. **a.** Continuous for all real numbers
b. Differentiable everywhere except at $x = 4$
c. $f(1) = 5$
d. $f'(1) = 0$ and $f'(3) = 0$
e. $f'(x) > 0$ on $(-\infty, 1)$ and $(4, \infty)$
f. $f'(x) < 0$ on $(1, 3)$ and $(3, 4)$
g. $\lim_{x \to 4^-} f'(x) = -\infty$ and $\lim_{x \to 4^+} f'(x) = \infty$
h. $f''(x) > 0$ on $(2, 3)$
i. $f''(x) < 0$ on $(-\infty, 2)$, $(3, 4)$, and $(4, \infty)$

40. On many calculators, graphs of rational functions produce lines at vertical asymptotes. For example, graphing $y = (x - 1)/(x + 1)$ on the window $[-4.9, 4.9]$ by $[-4.9, 4.9]$ produces such a line at $x = -1$ on the TI-84 Plus and TI-89. But with the window $[-4.7, 4.7]$ by $[-4.7, 4.7]$ on a TI-84 Plus; or $[-7.9, 7.9]$ by $[-7.9, 7.9]$ on a TI-89, the spurious line does not appear. Experiment with this function on your calculator, trying different windows, and try to figure out an explanation for this phenomenon. (*Hint:* Consider the number of pixels on the calculator screen.)

YOUR TURN ANSWERS

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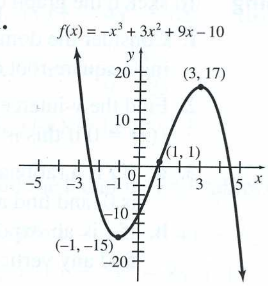

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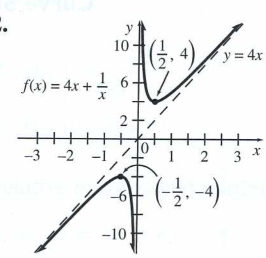

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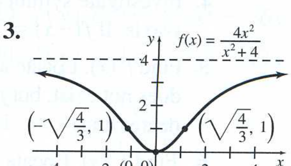

4.