1. Find four distinct matrix representations for the Klien Four group

$$V_4 = \langle a, b | a^2 = b^2 = (ab)^2 = 1 \rangle.$$

   Now translate your answer to the module case.

2. Find three different (distinct) matrix representations of the dihedral group of order 6.

3. Find Five distinct matrix representations of the dihedral group of order 8;

$$G = D_8 = \langle a, b | a^4 = b^2 = (ab)^2 = 1 \rangle$$

4. Let $D_{2n} = \langle a, b | a^n = b^2 = 1; ab = ba^{-1} \rangle$ be the dihedral group of order $2n$.
   (a) Show that

$$\rho : D_{2n} \to GL(2, \mathbb{C})$$

   such that:

$$\rho(a) = \begin{pmatrix} \bar{\xi} & 0 \\ 0 & \xi \end{pmatrix}$$

   and

$$\rho(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

such that:

$$\rho(a) = \begin{pmatrix} \bar{\xi} & 0 \\ 0 & \xi \end{pmatrix}$$

and

$$\rho(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a matrix representation of the dihedral group D_{2n}. Note that ξ is the nth root of unity; namely $\xi = e^{\frac{2\pi i}{n}}$.

(b) Decide if the representation $\rho : D_{2n} \longrightarrow GL(2, \mathbb{C})$ above is equivalent to the representation $\sigma : D_{2n} \longrightarrow GL(2, \mathbb{C})$ such that

$$\sigma(a) = \begin{pmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{pmatrix}$$

and

$$\sigma(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where α is the angle $2\pi/n$.

All representations are over the complex numbers $\mathcal{C}$ and all groups under considerations are finite.

1. Let G be a finite group. We say that $x,\ y \in G$ are conjugate in G if there is an element $g \in G$ such that $x^g = y$, where $x^g = g\cdot x\cdot g^{-1}$.
   (a) Show that this relation is an equivalence relation with equivalence classes which is called the conjugacy classes of G.
   (b) Show that the size of an orbit containing x is the index $[G : C_G(x)]$, where $C_G(x)$ is the centralizer of the element x.
   (c) Find the conjugacy classes of the following groups: V_4, D_6, S_3, $SL(2,2)$, D_8, A_4, S_4, $SL(2,3)$, $GL(2,3)$ and S_5.
2. Find four IRREDUCIBLE matrix representations for the Klien Four group

$$V_4 = \langle a, b | a^2 = b^2 = (ab)^2 = 1\rangle.$$

3. State and prove the converse of Schur's Lemma.
4. Let $D_{2n} = \langle a, b | a^n = b^2 = 1; ab = ba^{-1}\rangle$ be the dihedral group of order $2n$.

$$V_4 = \langle a, b | a^2 = b^2 = (ab)^2 = 1 \rangle.$$

3. State and prove the converse of Schur's Lemma.
4. Let $D_{2n} = \langle a, b | a^n = b^2 = 1; ab = ba^{-1} \rangle$ be the dihedral group of order $2n$. Use the converse of Schur's Lemma to show that

$$\rho : D_{2n} \longrightarrow GL(2, \mathcal{C})$$

such that:

$$\rho(a) = \begin{pmatrix} \bar{\xi} & 0 \\ 0 & \xi \end{pmatrix}$$

and

$$\rho(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is an IRREDUCIBLE matrix representation of the dihedral group D_{2n}. Note that ξ is the nth root of unity; namely $\xi = e^{\frac{2\pi i}{n}}$.

5. Find three IRREDUCIBLE representations of the dihedral group of order 6.
6. Find Five IRREDUCIBLE matrix representations of the dihedral group of order 8;

$$G = D_8 = \langle a, b | a^4 = b^2 = (ab)^2 = 1 \rangle$$

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7. Decompose the matrix representation

$$\rho : S_3 \to GL(3, C)$$

such that:

$$\rho((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\rho((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

into IRREDUCIBLE sub-representations of the symmetric group S_3.