Take Home Exam No. 2

Do 5 of the problems below:

#1 Suppose $V \in \mathcal{V}_F$ is a vector space over the field F and $|V| > 2$, show that $\exists$ more than one basis of V.

#2 Let F be a subfield of a field F', show that (i) F' is a vector space over F, (ii) If $\dim_F F' = m$, $V \in \mathcal{V}_{F'}$, $\dim_{F'} V = n$, Then V is a vector space over F of dimension $m \cdot n$.

#3 Let $V \in \mathcal{V}_F$ and $\dim_F V = n < \infty$, suppose $T \in L(V,V)$ is a linear map from V to V and for all $S \in L(V,V)$, $T \circ S = S \circ T$, show that $T = a I_V$, I_V the identity linear map of V into V, and $a \in F$.

#4 Let $V \in \mathcal{V}_F$, $\dim_F V = n < \infty$, let $T \in L(V,V)$ be such that range $T = \operatorname{Im} T =$ nullspace of $T = \ker T$, show that n is an even integer.

#5 Let $V \in \mathcal{V}_F$, $\dim_F V = n < \infty$, a subspace W of V of dimension $n-1$ is called a <u>hyperspace</u>. show that $\varphi \in V^* \Rightarrow \ker \varphi$ is a hyperspace if $\varphi \neq 0$.

#6 Let $V \in \mathcal{V}_F$, S a subset of V, The <u>annihilator</u> of S, denoted by S^0 $= \{\varphi \in V^* \mid \varphi(v) = 0 \ \forall v \in S\}$. show that if W is a subspace of V, Then $\dim_F W + \dim_F W^0 = \dim_F V$. Assume $\dim_F V = n < \infty$.

#7. If W_1, W_2 are subspaces of $V \in \mathcal{V}_F$, $\dim_F V = n < \infty$, Then $W_1 = W_2 \Longleftrightarrow W_1^0 = W_2^0$

#8. Suppose W is a finite dimensional subspace of $V \in \mathcal{V}_F$ such that V/W is finite dimensional show that V must be finite dimensional.

#9. Let F be a field, show that the vector space $V = F$ over F is not the direct sum of any two proper subspaces of F. Does F have a proper subspace?