Math 413/513

Take Home Exam. #1

Name:

Do 5 of the following nine problems.

#1. Let $W_1, W_2$ be submodules of an $R$-module $M \in \mathcal{M}_R$, suppose $W_1 \cup W_2$ is a submodule of $M$, show that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

#2. Let $K, L, M$ be submodules of a module $P \in \mathcal{M}_R$, suppose $K \supseteq L$ show that $K \cap (L+M) = L + (K \cap M)$

#3. Let $V \in \mathcal{V}_F$ be a vectorspace over the field $F$, let $T \in L(V,V)$ be a linear map, show that $T^2 = T \circ T = 0 \iff \exists$ subspaces $M, N$ of $V$ such that

a) $M + N = V$ b) $M \cap N = \{0\}$, (c) $TN = \{0\}$, $TM \subseteq N$.

#4. Let $V \in \mathcal{V}_F$ be a vectorspace over the field $F$, let $T \in L(V,V)$ be a linear map such that $T^2 = T \circ T = I_V$ (such a $T$ is called an involution). Show that if the characteristic of $F \neq 2$ (i.e. $1+1 \neq 0$ in $F$), then $\exists$ subspaces $M, N$ of $V$ with: (a) $M + N = V$, (b) $M \cap N = \{0\}$, (c) $T|_M = I_M$, (d) $T|_N = -I_N$.

#5. Let $V \in \mathcal{V}_F$, $T \in L(V,V)$, show that the following are equivalent:

(a) $\operatorname{Im} T \cap \ker T = \{0\}$, (b) If $T^2(v) = 0 \Rightarrow Tv = 0$, $v \in V$.

#6 Let $R$ be an integral domain (a ring $R$ such that $r_1 \cdot r_2 = 0 \Rightarrow r_1$ or $r_2 = 0$, $r_1, r_2 \in R$), let $M \in \mathcal{M}_R$, if $m \in M$ and $\exists r \in R$, $r \neq 0$ such that $r \cdot m = 0$, Then $m$ is called a Torsion element of $M$. Show That The set $M_{Tor} = \{ m \in M \mid m \text{ is a Torsion element of } M \}$ is a submodule of $M$. Also show That $M/M_{Tor}$ is Torsion free, That is, $\bar{0}$ is The only Torsion element of $M/M_{Tor}$.

#7 Let $T \in L(V,V)$, $V \in \mathcal{V}_F$ and suppose $T|_W = I_W$, $W$ a subspace of $V$

a) show That $T$ induces a linear map $S \in L(V/W, V/W)$

b) If The $S$ of (a) satisfies $S = I_{V/W}$, Then $U = T - I_V$ has The property $U^2 = U \circ U = 0$

#8 Let $V \in \mathcal{V}_F$, $T \in L(V,V)$. If $T^3 = T$, show That $V = V_0 \oplus V_1 \oplus V_2$, (i.e. $V = V_0 + V_1 + V_2$ and $V_i \cap (V_j + V_k) = \{0\}$, $i, j, k \in \{0, 1, 2\}$, $i \notin \{j, k\}$), where $V_i$ are subspaces of $V$, $i = 0, 1, 2$ such That $v \in V_0 \Rightarrow Tv = 0$, $v \in V_1 \Rightarrow Tv = v$ and $v \in V_2 \Rightarrow Tv = -v$. Assume That The characteristic of $F \neq 2$.

#9 Prove that a commutative ring with $1_R$ is a field $\Leftrightarrow$ The only ideals of $R$ are $\{0\}$ and $R$.