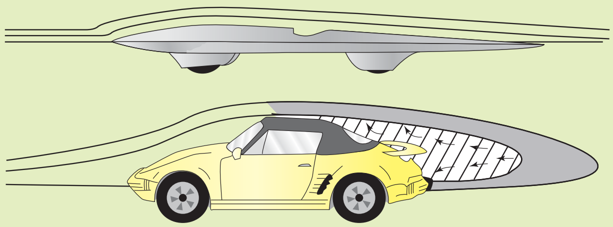
6 Factoring Polynomials and Solving Equations

What we see depends mainly on what we look for.

—John Lubbock

Most cars are designed so that their exteriors are curved and smooth. This characteristic is especially true for solar cars. In fact, a side view of a solar car often resembles the cross section of an airplane wing, as illustrated in the accompanying figure. This design reduces drag from air resistance and increases fuel efficiency.

Views of a Solar Car and a Standard Car



Mathematics plays an important role in the design of cars. Third-degree polynomials called *cubic splines* are used extensively by engineers to obtain a smooth shape for new cars. Although the topic of cubic splines is covered in advanced mathematics and engineering courses, in this chapter we introduce many concepts necessary for understanding polynomials and polynomial equations.

Source: R. Burden and J. Faires. *Numerical Analysis.*

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6.1 Introduction to Factoring

Basic Concepts • Common Factors • Factoring by Grouping

A LOOK INTO MATH ▶

Polynomials are frequently used in applications to approximate data and to model such things as changes in the weather, sales of a new video game, and the growth of young children. As a result, scientists and mathematicians commonly solve equations involving polynomials. One way to solve these equations is to use *factoring*. In this section we introduce two basic methods of factoring polynomials.

NEW VOCABULARY

- ☐ Factoring a polynomial
- ☐ Greatest common factor (GCF)

Basic Concepts

When two or more numbers are multiplied, each number is called a factor. For example, in the equation $4 \cdot 6 = 24$, the numbers 4 and 6 are factors and we say that 24 can be *factored* into the product $4 \cdot 6$. Other ways to factor 24 include

$$2 \cdot 12, \quad 8 \cdot 3, \quad 2 \cdot 2 \cdot 6, \quad \text{and} \quad 2 \cdot 2 \cdot 2 \cdot 3.$$

When we factor a positive number, we reverse the multiplication process and write the number as a product of two or more smaller numbers. Similarly, we **factor a polynomial** by writing the polynomial as a product of two or more *lower degree* polynomials. For example, possible ways to factor the polynomial $12x^5 + 6x^4$ include

$$2x(6x^4 + 3x^3), \quad x^2(12x^3 + 6x^2), \quad \text{and} \quad 6x^4(2x + 1).$$

The second of these factorizations shows that the **fifth**-degree polynomial $12x^5 + 6x^4$ can be written as the product of the **second**-degree polynomial x^2 and the **third**-degree polynomial $12x^3 + 6x^2$, where each factor has a lower degree than the given polynomial. Similar statements can be made about the first and third factorizations.

Common Factors

When factoring a polynomial, we first look for factors that are common to each term. By applying the distributive property we can often write a polynomial as a product. For example, each term in the polynomial $8x^2 + 6x$ has a factor of $2x$ because

$$8x^2 = 2x \cdot 4x \quad \text{and} \quad 6x = 2x \cdot 3.$$

Therefore by the distributive property,

$$8x^2 + 6x = 2x(4x + 3).$$

Thus the product $2x(4x + 3)$ equals $8x^2 + 6x$. We check this result by multiplying.

$$\begin{aligned} 2x(4x + 3) &= 2x \cdot 4x + 2x \cdot 3 \\ &= 8x^2 + 6x \ \checkmark \qquad \text{It checks.} \end{aligned}$$

This factorization is shown visually in Figure 6.1, where possible dimensions for a rectangle with an area of $8x^2 + 6x$ are $2x$ by $4x + 3$.

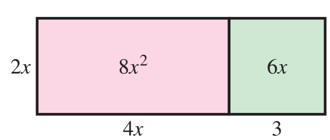

Figure 6.1 $8x^2 + 6x = 2x(4x + 3)$

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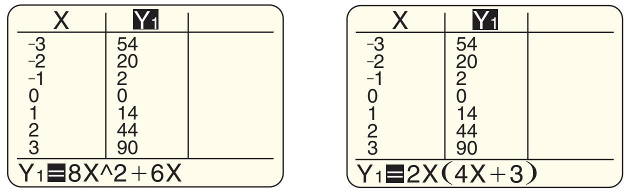
CALCULATOR HELP
To make a table of values, see Appendix A (pages AP-2 and AP-3).

The expressions $8x^2 + 6x$ and $2x(4x + 3)$ are equal for all values of x. Figure 6.2 illustrates this fact with a partial table of values for each expression.



(a) (b)

Figure 6.2 $8x^2 + 6x = 2x(4x + 3)$

EXAMPLE 1

Factoring an expression

Factor the expression and sketch a rectangle that illustrates the factorization.
(a) $10x + 6$ **(b)** $6x^2 + 15x$

Solution

(a) Each term in the polynomial $10x + 6$ has a factor of 2 because

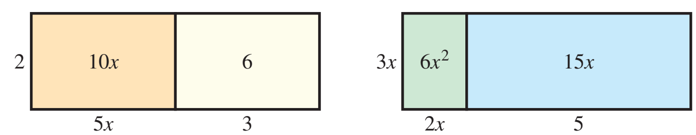
$$10x = 2 \cdot 5x \quad \text{and} \quad 6 = 2 \cdot 3.$$

By the distributive property, $10x + 6 = 2(5x + 3)$. This factorization is illustrated visually in Figure 6.3(a).

(b) Each term in the polynomial $6x^2 + 15x$ has a factor of $3x$ because

$$6x^2 = 3x \cdot 2x \quad \text{and} \quad 15x = 3x \cdot 5.$$

By the distributive property, $6x^2 + 15x = 3x(2x + 5)$. This factorization is illustrated visually in Figure 6.3(b).



(a) (b)

Figure 6.3

Now Try Exercises 9, 11

EXAMPLE 2

Finding common factors

Factor.
(a) $15x^2 + 10x$ **(b)** $6y^3 - 2y^2$ **(c)** $3z^3 + 9z^2 - 6z$ **(d)** $2x^2y^2 + 4xy^3$

Solution

(a) In the expression $15x^2 + 10x$, the terms $15x^2$ and $10x$ contain a common factor of $5x$.

$$15x^2 = 5x \cdot 3x \quad \text{and} \quad 10x = 5x \cdot 2$$

Therefore this polynomial can be factored as

$$15x^2 + 10x = 5x(3x + 2).$$

(b) In the expression $6y^3 - 2y^2$, the terms $6y^3$ and $2y^2$ contain a common factor of $2y^2$.

$$6y^3 = 2y^2 \cdot 3y \quad \text{and} \quad 2y^2 = 2y^2 \cdot 1$$

Therefore this polynomial can be factored as

$$6y^3 - 2y^2 = 2y^2(3y - 1).$$

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(c) In $3z^3 + 9z^2 - 6z$, the terms $3z^3$, $9z^2$, and $6z$ contain a common factor of $3z$.

$$3z^3 = 3z \cdot z^2, \quad 9z^2 = 3z \cdot 3z, \quad \text{and} \quad 6z = 3z \cdot 2$$

Therefore this polynomial can be factored as

$$3z^3 + 9z^2 - 6z = 3z(z^2 + 3z - 2).$$

(d) In $2x^2y^2 + 4xy^3$, the terms $2x^2y^2$ and $4xy^3$ contain a common factor of $2xy^2$.

$$2x^2y^2 = 2xy^2 \cdot x \quad \text{and} \quad 4xy^3 = 2xy^2 \cdot 2y$$

Thus $2x^2y^2 + 4xy^3 = 2xy^2(x + 2y)$.

Now Try Exercises 15, 17, 19, 21

In most situations we factor out the *greatest common factor* (GCF). For example, the polynomial $12b^3 + 8b^2$ has a common factor of $2b$. We could factor this polynomial as

$$12b^3 + 8b^2 = 2b(6b^2 + 4b).$$

However, we can factor out $4b^2$ instead.

$$12b^3 + 8b^2 = 4b^2(3b + 2)$$

Because $4b^2$ is the common factor with the greatest (integer) coefficient and highest degree, we say that $4b^2$ is the **greatest common factor** of $12b^3 + 8b^2$. In Examples 1 and 2 we factored out the greatest common factor for each expression.

When finding the greatest common factor for a polynomial, it is often helpful first to completely factor each term of the polynomial by writing its coefficient as the product of prime numbers and writing any powers of variables as repeated multiplication. For example, the complete factorization of $24x^4$ is $2 \cdot 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot x \cdot x$, and the complete factorization of $18xy^3$ is $2 \cdot 3 \cdot 3 \cdot x \cdot y \cdot y \cdot y$. The next example demonstrates this process.

READING CHECK

- How do you know if a common factor is actually the greatest common factor?

STUDY TIP

In general, the ability to factor polynomials is important in solving many types of equations. Factoring out the GCF is a first step in preparing a polynomial for other factoring techniques discussed in this chapter.

EXAMPLE 3

Finding the greatest common factor

Find the greatest common factor for each expression. Then factor the expression.
(a) $9x^2 + 6x$ **(b)** $4z^4 + 8z^2$ **(c)** $8a^2b^3 - 16a^3b^2$

Solution

(a) Because

$$9x^2 = 3 \cdot 3 \cdot x \cdot x \quad \text{and} \quad 6x = 3 \cdot 2 \cdot x,$$

the terms have common factors of **3** and x. The GCF is the product of these two factors, or $3 \cdot x = 3x$. Thus the expression $9x^2 + 6x$ can be factored as $3x(3x + 2)$. Note that the term $3x$ inside the parentheses is the product of the (black) factors of $9x^2$ that were not part of the greatest common factor, and the term 2 inside the parentheses is the product of the (black) factors of $6x$ that were not part of the greatest common factor.

(b) Because

$$4z^4 = 2 \cdot 2 \cdot z \cdot z \cdot z \cdot z \quad \text{and}$$
$$8z^2 = 2 \cdot 2 \cdot 2 \cdot z \cdot z,$$

the terms have common factors of **2**, **2**, z, and z. The GCF is the product of these four factors, or $2 \cdot 2 \cdot z \cdot z = 4z^2$. Thus $4z^4 + 8z^2$ can be factored as $4z^2(z^2 + 2)$.

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(c) Because

$$8a^2b^3 = 2 \cdot 2 \cdot 2 \cdot a \cdot a \cdot b \cdot b \cdot b \quad \text{and}$$
$$16a^3b^2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot a \cdot a \cdot a \cdot b \cdot b,$$

the terms have common factors of 2, 2, 2, a, a, b, and b. The GCF is the product of these seven factors, or $2 \cdot 2 \cdot 2 \cdot a \cdot a \cdot b \cdot b = 8a^2b^2$. Here $8a^2b^3 - 16a^3b^2$ can be factored as $8a^2b^2(b - 2a)$.

Now Try Exercises 23, 25, 37

NOTE: With practice, you may find that you can determine the GCF mentally without factoring each term as was done in Example 3.

MAKING CONNECTIONS

Checking Common Factors with Multiplication

When factoring we can check our work by multiplying. For example, if we are uncertain whether the equation

$$6y^3 - 2y^2 = 2y^2(3y - 1)$$

is correct, we can apply the distributive property to the right side of the above equation to obtain

$$2y^2(3y - 1) = 2y^2 \cdot 3y - 2y^2 \cdot 1$$
$$= 6y^3 - 2y^2. \checkmark \qquad \text{It checks.}$$

In the next example, we factor an expression that occurs in a scientific application.

EXAMPLE 4 **Modeling the flight of a golf ball**

If a golf ball is hit upward at 66 feet per second (45 miles per hour), then its height in feet after t seconds is approximated by $66t - 16t^2$. Factor this expression.

Solution

The GCF for $66t$ and $16t^2$ is $2t$ because

$$66t = 2t \cdot 33 \quad \text{and} \quad 16t^2 = 2t \cdot 8t.$$

Therefore this polynomial can be factored as

$$66t - 16t^2 = 2t(33 - 8t).$$

Now Try Exercise 81

Factoring by Grouping

Factoring by grouping is a technique that makes use of the associative and distributive properties. The next example illustrates one step in this factoring technique.

EXAMPLE 5 **Factoring out binomials**

Factor.

(a) $5x(x + 3) + 6(x + 3)$ **(b)** $x^2(2x - 5) - 4x(2x - 5)$

Solution

(a) Each term in the expression $5x(x + 3) + 6(x + 3)$ contains the binomial $(x + 3)$. Therefore the distributive property can be used to factor this expression.

$$5x(x + 3) + 6(x + 3) = (5x + 6)(x + 3)$$

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(b) Each term in $x^2(2x - 5) - 4x(2x - 5)$ contains the binomial $(2x - 5)$. Therefore the distributive property can be used to factor this expression.

$$x^2(2x - 5) - 4x(2x - 5) = (x^2 - 4x)(2x - 5)$$
$$= x(x - 4)(2x - 5)$$

Now Try Exercises 41, 45

Now consider the polynomial

$$x^3 + x^2 + 2x + 2.$$

We can factor this polynomial by first grouping it into two binomials.

$(x^3 + x^2) + (2x + 2)$ Associative property

$x^2(x + 1) + 2(x + 1)$ Factor out common factors.

$(x^2 + 2)(x + 1)$ Factor out $(x + 1)$.

When factoring by grouping, we factor out a common factor more than once.

The first step in factoring a four-term polynomial by grouping requires the use of the associative property to write the polynomial as the *sum* of two binomials. However, this property must be applied carefully to avoid sign errors. The next two examples illustrate that the middle arithmetic symbol (+ or −) in a four-term polynomial determines how the associative property is applied.

READING CHECK

- What factoring technique is sometimes used to factor four-term polynomials?

EXAMPLE 6 **Factoring by grouping when the middle symbol is (+)**

Factor each polynomial.

(a) $2x^3 - 4x^2 + 3x - 6$ **(b)** $3x + 3y + ax + ay$

Solution

(a) Use the associative property to write the polynomial as the *sum* of two binomials.

$2x^3 - 4x^2 + 3x - 6 = (2x^3 - 4x^2) + (3x - 6)$ Associative property

$= 2x^2(x - 2) + 3(x - 2)$ Factor out common factors.

$= (2x^2 + 3)(x - 2)$ Factor out $(x - 2)$.

(b) Group the polynomial into the *sum* of two binomials.

$3x + 3y + ax + ay = (3x + 3y) + (ax + ay)$ Associative property

$= 3(x + y) + a(x + y)$ Factor out common factors.

$= (3 + a)(x + y)$ Factor out $(x + y)$.

Now Try Exercises 47, 65

EXAMPLE 7 **Factoring by grouping when the middle symbol is (−)**

Factor each polynomial.

(a) $3y^3 - y^2 - 9y + 3$ **(b)** $z^3 + 4z^2 - 5z - 20$

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Solution

(a) Begin by changing the middle subtraction to addition by adding the opposite of $9y$. Then apply the associative property to write the result as the *sum* of two binomials.

$$\begin{aligned} 3y^3 - y^2 - 9y + 3 &= 3y^3 - y^2 + (-9y) + 3 && \text{Add the opposite of } 9y. \\ &= (3y^3 - y^2) + (-9y + 3) && \text{Associative property} \\ &= y^2(3y - 1) - 3(3y - 1) && \text{Factor out common factors.} \\ &= (y^2 - 3)(3y - 1) && \text{Factor out } (3y - 1). \end{aligned}$$

Note that in the third step, -3 was factored from the second binomial.

(b) Begin by changing the middle subtraction to addition by adding the opposite of $5z$. Then apply the associative property to write the result as the *sum* of two binomials.

$$\begin{aligned} z^3 + 4z^2 - 5z - 20 &= z^3 + 4z^2 + (-5z) - 20 && \text{Add the opposite of } 5z. \\ &= (z^3 + 4z^2) + (-5z - 20) && \text{Associative property} \\ &= z^2(z + 4) - 5(z + 4) && \text{Factor out common factors.} \\ &= (z^2 - 5)(z + 4) && \text{Factor out } (z + 4). \end{aligned}$$

Now Try Exercises 55, 61

When factoring some polynomials, it may be necessary to factor out the greatest common factor before completing other factoring techniques such as factoring by grouping. In the next example the GCF is factored out before grouping is applied.

EXAMPLE 8 **Factoring out the GCF before grouping**

Completely factor each polynomial.

(a) $6x^3 - 12x^2 - 3x + 6$
(b) $2x^5 - 8x^4 + 6x^3 - 24x^2$

Solution

(a) The GCF of $6x^3 - 12x^2 - 3x + 6$ is 3, so factor out **3** before factoring the remaining polynomial by grouping.

$$\begin{aligned} 6x^3 - 12x^2 - 3x + 6 &= 3(2x^3 - 4x^2 - x + 2) && \text{Factor out the GCF.} \\ &= 3(2x^3 - 4x^2 + (-x) + 2) && \text{Add the opposite of } x. \\ &= 3\big((2x^3 - 4x^2) + (-x + 2)\big) && \text{Associative property} \\ &= 3\big(2x^2(x - 2) - 1(x - 2)\big) && \text{Factor out common factors.} \\ &= 3(2x^2 - 1)(x - 2) && \text{Factor out } (x - 2). \end{aligned}$$

(b) The GCF of $2x^5 - 8x^4 + 6x^3 - 24x^2$ is $2x^2$, so factor out $\mathbf{2x^2}$ before factoring the remaining polynomial by grouping.

$$\begin{aligned} 2x^5 - 8x^4 + 6x^3 - 24x^2 &= 2x^2(x^3 - 4x^2 + 3x - 12) && \text{Factor out the GCF.} \\ &= 2x^2\big((x^3 - 4x^2) + (3x - 12)\big) && \text{Associative property} \\ &= 2x^2\big(x^2(x - 4) + 3(x - 4)\big) && \text{Factor out common factors.} \\ &= 2x^2(x^2 + 3)(x - 4) && \text{Factor out } (x - 4). \end{aligned}$$

Now Try Exercises 67, 71

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6.1 Putting It All Together

CONCEPT	EXPLANATION	EXAMPLES
Common Factor	Factor out a monomial common to each term in a polynomial.	$6z^2 - 6z = 6z(z - 1)$ $4y^3 - 6y^2 = 2y^2(2y - 3)$ $5x^3 - 10x^2 + 15x = 5x(x^2 - 2x + 3)$ $2a^3b^3 - 4a^2b^3 = 2a^2b^3(a - 2)$
Greatest Common Factor (GCF)	The common factor with the greatest (integer) coefficient and highest degree	The GCF of $10x^4 + 15x^2$ is $5x^2$. Common factors include 1, 5, x, $5x$, x^2, and $5x^2$. However, $5x^2$ is *the greatest* common factor.
Factoring by Grouping	Factoring by grouping is a method that can be used to factor some *four-term* polynomials into a product of two binomials. It makes use of the associative and distributive properties.	$2x^3 + 3x^2 + 2x + 3$ $= (2x^3 + 3x^2) + (2x + 3)$ $= x^2(2x + 3) + 1(2x + 3)$ $= (x^2 + 1)(2x + 3)$ $4x^3 - 24x^2 - 3x + 18$ $= 4x^3 - 24x^2 + (-3x) + 18$ $= (4x^3 - 24x^2) + (-3x + 18)$ $= 4x^2(x - 6) - 3(x - 6)$ $= (4x^2 - 3)(x - 6)$

6.1 Exercises

CONCEPTS AND VOCABULARY

1. To _____ a polynomial, write it as a product of two or more lower degree polynomials.

2. A common factor in the expression $ab + ac$ is _____.

3. When factoring, we can check our work by _____.

4. When finding the GCF for a polynomial, it is often helpful to completely factor each term by writing its coefficient as the product of _____ numbers and writing any powers of variables as repeated _____.

5. The _____ of a polynomial is the common factor with the greatest (integer) coefficient and highest degree.

6. Factoring by _____ can be used to factor some four-term polynomials into a product of two binomials by using the associative and distributive properties.

7. Identify four common factors of $2x^2 + 4x$.

8. Identify the greatest common factor (GCF) of the expression $2x^2 + 4x$.

COMMON FACTORS

Exercises 9–14: Factor the expression. Then make a sketch of a rectangle that illustrates this factorization.

9. $2x + 4$
10. $6 + 3x$
11. $z^2 + 4z$
12. $a^2 + 5a$
13. $3y^2 + 12y$
14. $2z^2 + 10z$

Exercises 15–22: Factor the expression.

15. $3x^2 + 9x$
16. $10y^2 + 2y$

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17. $4y^3 - 2y^2$

18. $6x^4 + 9x^2$

19. $2z^3 + 8z^2 - 4z$

20. $5x^4 - 15x^3 - 10x^2$

21. $6x^2y - 3xy^2$

22. $7x^3y^3 + 14x^2y^2$

Exercises 23–40: Identify the greatest common factor. Then factor the expression.

23. $6x - 18x^2$

24. $16x^2 - 24x^3$

25. $8y^3 - 12y^2$

26. $12y^3 - 8y^2 + 4y$

27. $6z^3 + 3z^2 + 9z$

28. $16z^3 - 24z^2 - 36z$

29. $x^4 - 5x^3 - 4x^2$

30. $2x^4 + 8x^2$

31. $5y^5 + 10y^4 - 15y^3 + 10y^2$

32. $7y^4 - 14y^3 - 21y^2 + 7y$

33. $xy + xz$

34. $ab - bc$

35. $ab^2 - a^2b$

36. $4x^2y + 6xy^2$

37. $5x^2y^4 + 10x^3y^3$

38. $3r^3t^3 - 6r^4t^2$

39. $a^2b + ab^2 + ab$

40. $6ab^2 - 9ab + 12b^2$

FACTORING BY GROUPING

Exercises 41–46: Factor.

41. $x(x + 1) - 2(x + 1)$

42. $5x(3x - 2) + 2(3x - 2)$

43. $(z + 5)z + (z + 5)4$

44. $3y^2(y - 2) + 5(y - 2)$

45. $4x^3(x - 5) - 2x(x - 5)$

46. $8x^2(x + 3) + (x + 3)$

Exercises 47–66: Factor by grouping.

47. $x^3 + 2x^2 + 3x + 6$

48. $x^3 + 6x^2 + x + 6$

49. $2y^3 + y^2 + 2y + 1$

50. $4y^3 + 10y^2 + 2y + 5$

51. $2z^3 - 6z^2 + 5z - 15$

52. $15z^3 - 5z^2 + 6z - 2$

53. $4t^3 - 20t^2 + 3t - 15$

54. $4t^3 - 12t^2 + 3t - 9$

55. $9r^3 + 6r^2 - 6r - 4$

56. $3r^3 + 12r^2 - 2r - 8$

57. $7x^3 + 21x^2 - 2x - 6$

58. $6x^3 + 3x^2 - 10x - 5$

59. $2y^3 - 7y^2 - 4y + 14$

60. $y^3 - 5y^2 - 3y + 15$

61. $z^3 - 4z^2 - 7z + 28$

62. $12z^3 - 18z^2 - 10z + 15$

63. $2x^4 - 3x^3 + 4x - 6$

64. $x^4 + x^3 + 5x + 5$

65. $ax + bx + ay + by$

66. $ax - bx + ay - by$

Exercises 67–78: Completely factor the polynomial.

67. $3x^3 + 6x^2 + 3x + 6$

68. $5x^3 - 5x^2 + 5x - 5$

69. $6y^4 - 24y^3 - 2y^2 + 8y$

70. $6x^4 - 12x^3 + 3x^2 - 6x$

71. $x^5 + 2x^4 - 3x^3 - 6x^2$

72. $y^6 + 3y^5 - 2y^4 - 6y^3$

73. $4x^5 + 2x^4 - 12x^3 - 6x^2$

74. $18y^5 + 27y^4 + 12y^3 + 18y^2$

75. $x^3y + x^2y^2 - 2x^2y - 2xy^2$

76. $6x^3y - 3x^2y^2 + 18x^2y - 9xy^2$

77. $2x^3y^3 - 2x^4y^2 + 4x^2y^3 - 4x^3y^2$

78. $4x^2y^6 - 4x^2y^5 - 8xy^7 + 8xy^6$

79. **Thinking Generally** Factor a from the polynomial expression $ax^2 + bx + c$.

80. **Thinking Generally** Factor c from the polynomial expression $ax^2 + bx + c$.

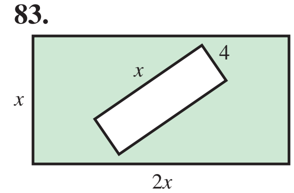
APPLICATIONS

81. ***Flight of a Golf Ball*** The height of a golf ball in feet after t seconds is given by $80t - 16t^2$.
(a) Identify the greatest common factor.
(b) Factor this expression.

82. ***Flight of a Golf Ball*** Repeat the previous exercise if the height of a golf ball in feet after t seconds is given by $128t - 16t^2$.

Exercises 83–86: ***Geometry*** *Use the information in the figure to write a polynomial expression that represents the area of the shaded region. Then factor the expression.*

83.



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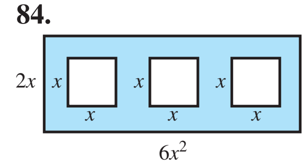

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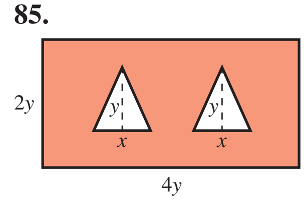

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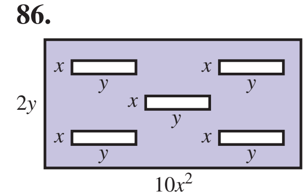

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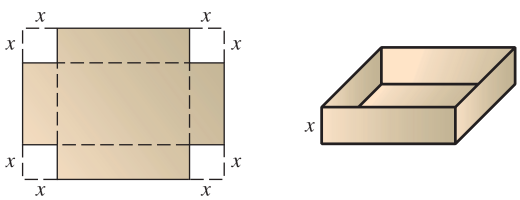
87. ***Volume of a Box*** A box is constructed by cutting out square corners of a rectangular piece of cardboard and folding up the sides. If the cutout corners have sides with length x, then the volume of the box is given by the polynomial $4x^3 - 60x^2 + 200x$.



(a) Find the volume of the box when $x = 3$ inches.
(b) Factor out the greatest common factor for this expression.

88. ***Volume of a Box*** (Refer to the preceding exercise.) A box is constructed from a square piece of metal that is 20 inches on a side.
(a) If the square corners of length x are cut out, write a polynomial that gives the volume of the box.
(b) Evaluate the polynomial when $x = 4$ inches.
(c) Factor out the greatest common factor for this polynomial expression.

WRITING ABOUT MATHEMATICS

89. Use an example to explain the difference between a common factor and the greatest common factor.

90. Use an example to explain how to factor a polynomial by grouping. What two properties of real numbers did you use?

6.2 Factoring Trinomials I ($x^2 + bx + c$)

Review of the FOIL Method • Factoring Trinomials with Leading Coefficient 1

A LOOK INTO MATH ▶

When astronauts are in training, they don't need to leave Earth's atmosphere to experience zero gravity. A specially modified jet following a precise curved path can provide about 30 seconds of weightlessness. The curved path can be modeled by a second degree polynomial with three terms (a trinomial). Specific information about the flight can be obtained by solving equations involving this trinomial. In this section, we discuss techniques for factoring trinomials with a *leading coefficient* of 1. (*Source:* NASA.)

NEW VOCABULARY

- ☐ Leading coefficient
- ☐ Prime polynomial

Review of the FOIL Method

Recall that a *trinomial* is a polynomial that has three terms. We begin by reviewing products of binomials that result in trinomials. For example, we multiply the binomials $(x + 2)$ and $(x + 3)$ as follows.

$$(x + 2)(x + 3) = x \cdot x + x \cdot 3 + 2 \cdot x + 2 \cdot 3$$
$$= x^2 + 5x + 6$$

Note that the first term, x^2, in the trinomial results from multiplying the *first* terms of each binomial. The middle term, $5x$, results from adding the product of the *outside* terms and the product of the *inside* terms. Finally, the last term, 6, results from multiplying the *last* terms of each binomial. We discussed this method of multiplying binomials, called FOIL, in Section 5.3 and illustrate it as follows.

F *L*

$$(x + 2)(x + 3) = x^2 + 5x + 6$$

I → $2x$
O → $+3x$
$5x$ ✓ ⟵ The middle term checks.

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Factoring Trinomials with Leading Coefficient 1

Any trinomial of degree 2 in the variable x can be written in *standard form* as $ax^2 + bx + c$, where a, b, and c are constants. The constant a is called the **leading coefficient** of the trinomial. In this section we focus on trinomials where $a = 1$ and b and c are integers.

Recall that the binomials $(x + m)$ and $(x + n)$ are multiplied as follows.

$$\begin{aligned}(x + m)(x + n) &= x^2 + nx + mx + mn \\ &= x^2 + (m + n)x + mn\end{aligned}$$

Note that the coefficient of the x-term is the sum of m and n and that the constant (or third) term is the product of m and n. Thus to factor a trinomial in the form $x^2 + bx + c$, we start by finding two numbers, m and n, such that when they are multiplied $m \cdot n = c$ and when they are added $m + n = b$. In the next example, we find integer pairs that have a specified product and sum.

EXAMPLE 1

Finding integer pairs with a given product and sum

For each of the following, find an integer pair that has the given product and sum.
(a) Product: 18; Sum: 11 **(b)** Product: –20; Sum: 1

Solution

(a) For two integers to have a product of (positive) 18, the integers must have the same sign. Because the specified sum is positive and any sum of two negative integers is negative, the two integers must be positive. Table 6.1 lists positive integer factor pairs for **18** along with the corresponding sum for each pair.

TABLE 6.1 Factor Pairs for 18

Factors	1, 18	**2, 9**	3, 6
Sum	19	**11**	9

From the table, we see that the integers **2** and **9** have a product of **18** and a sum of **11**.

(b) For two integers to have a negative product, they must have unlike signs. Integer factor pairs for **−20** and the corresponding sum for each pair are listed in Table 6.2.

TABLE 6.2 Factor Pairs for −20

Factors	1, −20	2, −10	4, −5	−1, 20	−2, 10	**−4, 5**
Sum	−19	−8	−1	19	8	**1**

Here we see that the integers **−4** and **5** have a product of **−20** and a sum of **1**.

Now Try Exercises 5, 9

STUDY TIP

Spend extra time practicing the process illustrated in Example 1. Pay special attention to factor pairs in which at least one of the integers is negative.

The ability to find integer pairs as demonstrated in Example 1 is essential for factoring trinomials of the form $x^2 + bx + c$. To illustrate the factoring process, we will find an integer pair that can be used to factor $x^2 + 6x + 8$.

Standard Form	***Example***
$x^2 + bx + c$	$x^2 + 6x + 8$
$m \cdot n = c$	$m \cdot n = 8$
$m + n = b$	$m + n = 6$

To determine possible values for m and n, we list factor pairs for 8 and search for a pair whose sum is 6, as in Table 6.3.

TABLE 6.3
Factor Pairs for 8

Factors	1, 8	2, 4
Sum	9	6

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Because $2 \cdot 4 = 8$ and $2 + 4 = 6$, we let $m = 2$ and $n = 4$. We then factor the given trinomial as

$$x^2 + 6x + 8 = (x + 2)(x + 4).$$

Note that, if you can find this factor pair mentally, making a table is not necessary.

We check the result by multiplying the two binomials.

$$(x + 2)(x + 4) = x^2 + 6x + 8$$

$$\begin{array}{r} 2x \\ +4x \\ \hline 6x \end{array} \checkmark$$

The middle term checks.

FACTORING $x^2 + bx + c$

To factor the trinomial $x^2 + bx + c$, find two numbers m and n that satisfy

$$m \cdot n = c \quad \text{and} \quad m + n = b.$$

Then $x^2 + bx + c = (x + m)(x + n)$.

EXAMPLE 2 **Factoring a trinomial having only positive coefficients**

Factor each trinomial.

(a) $x^2 + 7x + 12$ **(b)** $x^2 + 13x + 30$ **(c)** $z^2 + 9z + 20$

Solution

(a) To factor $x^2 + 7x + 12$ we need to find a factor pair for 12 whose sum is 7. To do so we make Table 6.4.

TABLE 6.4 Factor Pairs for 12

Factors	1, 12	2, 6	3, 4
Sum	13	8	7

The required factor pair is 3 and 4 because $3 \cdot 4 = 12$ and $3 + 4 = 7$. Therefore the given trinomial can be factored as

$$x^2 + 7x + 12 = (x + 3)(x + 4).$$

(b) To factor $x^2 + 13x + 30$ we need to find a factor pair for 30 whose sum is 13. The required pair is 3 and 10. Thus

$$x^2 + 13x + 30 = (x + 3)(x + 10).$$

(c) To factor $z^2 + 9z + 20$ we need to find a factor pair for 20 whose sum is 9. The required pair is 4 and 5. Thus

$$z^2 + 9z + 20 = (z + 4)(z + 5).$$

Now Try Exercises 19, 21, 23

In the next example, the coefficients of the middle terms are negative.

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EXAMPLE 3 **Factoring trinomials having a negative middle coefficient**

Factor each trinomial.

(a) $x^2 - 7x + 10$ **(b)** $x^2 - 8x + 15$ **(c)** $y^2 - 9y + 18$

Solution

(a) To factor $x^2 - 7x + 10$ we need to find a factor pair for 10 whose sum equals -7. To have a positive product *and* a negative sum, *both* numbers must be negative, as shown in Table 6.5.

TABLE 6.5 Factor Pairs for 10

Factors	$-1, -10$	$-2, -5$
Sum	-11	-7

The required pair is -2 and -5 because $-2 \cdot (-5) = 10$ and $-2 + (-5) = -7$. Therefore the given trinomial can be factored as

$$x^2 - 7x + 10 = (x - 2)(x - 5).$$

(b) To factor $x^2 - 8x + 15$ we need to find a factor pair for 15 whose sum is -8. The required pair is -3 and -5. Thus

$$x^2 - 8x + 15 = (x - 3)(x - 5).$$

(c) To factor $y^2 - 9y + 18$ we need to find a factor pair for 18 whose sum is -9. The required pair is -3 and -6. Thus

$$y^2 - 9y + 18 = (y - 3)(y - 6).$$

Now Try Exercises 27, 29, 31

In Examples 2 and 3 the coefficient of the last term was always positive. In the next example, this coefficient is negative and the coefficient of the middle term is either positive or negative.

EXAMPLE 4 **Factoring trinomials having a negative constant term**

Factor each trinomial.

(a) $x^2 - 3x - 4$ **(b)** $x^2 + 7x - 8$ **(c)** $t^2 - 2t - 24$

Solution

(a) To factor $x^2 - 3x - 4$ we need to find a factor pair for -4 whose sum is -3. To have a negative product one factor must be positive and the other factor must be negative, as shown in Table 6.6.

TABLE 6.6 Factor Pairs for -4

Factors	$-1, 4$	$1, -4$	$-2, 2$
Sum	3	-3	0

The required pair is 1 and -4 because $1 \cdot (-4) = -4$ and $1 + (-4) = -3$. Therefore the given trinomial can be factored as

$$x^2 - 3x - 4 = (x + 1)(x - 4),$$

which can be checked by multiplying $(x + 1)(x - 4)$.

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(b) To factor $x^2 + 7x - 8$ we need to find a factor pair for -8 whose sum is 7. The required pair is -1 and 8. Thus

$$x^2 + 7x - 8 = (x - 1)(x + 8).$$

(c) To factor $t^2 - 2t - 24$ we need to find a factor pair for -24 whose sum is -2. The required pair is -6 and 4. Thus

$$t^2 - 2t - 24 = (t - 6)(t + 4).$$

Now Try Exercises 37, 53

READING CHECK

- What is a prime polynomial?

A polynomial with integer coefficients that cannot be factored by using integer coefficients is called a **prime polynomial**. The next example illustrates that some trinomials of the form $x^2 + bx + c$ cannot be factored into the product of two binomials.

EXAMPLE 5 Discovering that a trinomial is prime

Factor each trinomial, if possible.

(a) $x^2 + 9x + 12$ **(b)** $x^2 + 5x - 4$

Solution

(a) To factor $x^2 + 9x + 12$, we need to find a factor pair for 12 whose sum is 9. Table 6.7 reveals that no such factor pair exists.

TABLE 6.7 Factor Pairs for 12

Factors	1, 12	2, 6	3, 4
Sum	13	8	7

The trinomial $x^2 + 9x + 12$ is prime.

(b) At first glance it may appear that the required factor pair is 4 and 1 because $4 \cdot 1 = 4$ and $4 + 1 = 5$. However, it is important to pay close attention to the signs of the coefficients. To factor $x^2 + 5x - 4$, we need to find a factor pair for -4 whose sum is 5. No such factor pair exists. The trinomial $x^2 + 5x - 4$ is prime.

Now Try Exercises 17, 41

READING CHECK

- How can you use the signs of the coefficients in a trinomial to determine the signs in the binomial factors?

MAKING CONNECTIONS

The Signs in the Binomial Factors

If a trinomial of the form $x^2 + bx + c$ can be factored, the signs of the coefficients in the trinomial can be used to determine the signs in the binomial factors. If c is positive, the binomial factors must have the same signs. If c is negative, the binomial factors must have opposite signs. If b and c represent positive numbers, this can be summarized as follows.

Form of the Trinomial	*Signs in the Binomial Factors*
$x^2 + bx + c$	$(+)(+)$
$x^2 - bx + c$	$(-)(-)$
$x^2 + bx - c$	$(-)(+)$
$x^2 - bx - c$	$(-)(+)$

When factoring some trinomials, it may be necessary to factor out the greatest common factor before attempting to factor the trinomial into the product of two binomials. The next example illustrates this process.

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EXAMPLE 6 **Factoring out the GCF before factoring further**

Factor each trinomial completely.
(a) $7x^2 + 35x + 42$ **(b)** $2x^4 - 4x^3 - 6x^2$

Solution

(a) Because the GCF of $7x^2 + 35x + 42$ is 7, factor out **7** first.

$$7x^2 + 35x + 42 = 7(x^2 + 5x + 6)$$

Now, to factor $x^2 + 5x + 6$, we need to find a factor pair for 6 whose sum is 5. The required pair is 2 and 3. Thus

$$7x^2 + 35x + 42 = 7(x + 2)(x + 3).$$

(b) Because the GCF of $2x^4 - 4x^3 - 6x^2$ is $2x^2$, factor out $\mathbf{2x^2}$ first.

$$2x^4 - 4x^3 - 6x^2 = 2x^2(x^2 - 2x - 3)$$

Now, to factor $x^2 - 2x - 3$, we need to find a factor pair for -3 whose sum is -2. The required pair is -3 and 1. Thus

$$2x^4 - 4x^3 - 6x^2 = 2x^2(x - 3)(x + 1).$$

Now Try Exercises 59, 71

CRITICAL THINKING

A cube has a surface area of $6x^2 + 24x + 24$. What is the length of each side?

EXAMPLE 7 **Finding the dimensions of a rectangle**

Find one possibility for the dimensions of a rectangle that has an area of $x^2 + 6x + 5$.

Solution

The area of a rectangle equals length times width. If we can factor $x^2 + 6x + 5$, then the factors can represent its length and width. Because

$$x^2 + 6x + 5 = (x + 1)(x + 5),$$

one possibility for the rectangle's dimensions is width $x + 1$ and length $x + 5$, as illustrated in Figure 6.4.

Now Try Exercise 87

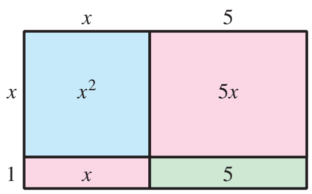

Figure 6.4 Area $= x^2 + 6x + 5$

6.2 Putting It All Together

CONCEPT	EXPLANATION	EXAMPLES
Factoring Trinomials of the Form $x^2 + bx + c$	Find two numbers m and n that satisfy $mn = c$ and $m + n = b$. Then $x^2 + bx + c = (x + m)(x + n)$.	$x^2 + 9x + 20 = (x + 4)(x + 5)$ because $4 \cdot 5 = 20$ and $4 + 5 = 9$. $x^2 + x - 6 = (x - 2)(x + 3)$ because $-2 \cdot 3 = -6$ and $-2 + 3 = 1$. $x^2 - 8x + 12 = (x - 6)(x - 2)$ because $-6 \cdot (-2) = 12$ and $-6 + (-2) = -8$.

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6.2 Exercises

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CONCEPTS AND VOCABULARY

1. In the trinomial $x^2 + bx + c$, the leading coefficient is _____.
2. Multiply $(x + m)(x + n)$. What is the coefficient of the x-term? What is the constant term?
3. To factor $x^2 + bx + c$, start by finding two numbers m and n that satisfy _____ $= c$ and _____ $= b$.
4. A trinomial with integer coefficients that cannot be factored using integer coefficients is _____.

PRODUCTS AND SUMS

Exercises 5–12: Find the integer pair that has the given product and sum.

5. Product: 28 Sum: 11
6. Product: 35 Sum: 12
7. Product: −30 Sum: −7
8. Product: −15 Sum: −2
9. Product: −50 Sum: 5
10. Product: −100 Sum: 21
11. Product: 28 Sum: −11
12. Product: 80 Sum: −42

FACTORING TRINOMIALS

Exercises 13–58: Factor the trinomial. If the trinomial cannot be factored, write "prime."

13. $x^2 + 3x + 2$
14. $x^2 + 5x + 4$
15. $y^2 + 4y + 4$
16. $y^2 + 8y + 7$
17. $z^2 + 3z + 7$
18. $z^2 + 4z + 5$
19. $x^2 + 8x + 15$
20. $x^2 + 9x + 14$
21. $m^2 + 13m + 36$
22. $m^2 + 15m + 36$
23. $n^2 + 20n + 100$
24. $n^2 + 52n + 100$
25. $x^2 - 6x + 5$
26. $x^2 - 6x + 8$
27. $y^2 - 7y + 12$
28. $y^2 - 12y + 27$
29. $z^2 - 13z + 40$
30. $z^2 - 15z + 54$
31. $a^2 - 16a + 63$
32. $a^2 - 82a + 81$
33. $y^2 - 6y + 10$
34. $y^2 - 2y + 3$
35. $b^2 - 30b + 125$
36. $b^2 - 19b + 90$
37. $x^2 + 13x - 90$
38. $x^2 + 15x - 100$
39. $m^2 + 4m - 45$
40. $m^2 + 4m - 60$
41. $a^2 + 16a - 63$
42. $a^2 + 13a - 42$
43. $n^2 + 10n - 200$
44. $n^2 + 2n - 120$
45. $x^2 + 22x - 23$
46. $x^2 + 18x - 19$
47. $a^2 + 4a - 32$
48. $a^2 + 9a - 36$
49. $b^2 - b - 20$
50. $b^2 - b - 12$
51. $m^2 - 14m - 22$
52. $m^2 - 11m - 24$
53. $x^2 - x - 72$
54. $x^2 - 2x - 80$
55. $y^2 - 15y - 34$
56. $y^2 - 10y - 39$
57. $z^2 - 5z - 66$
58. $z^2 - 6z - 55$

Exercises 59–74: Factor the trinomial completely.

59. $5x^2 - 10x - 40$
60. $2x^2 + 8x - 10$
61. $-3m^2 - 9m + 12$
62. $-4n^2 + 20n - 24$
63. $y^3 - 7y^2 + 10y$
64. $z^3 + 9z^2 + 20z$
65. $-x^3 - 2x^2 + 15x$
66. $-y^3 + 9y^2 - 14y$
67. $3a^3 + 21a^2 + 18a$
68. $5b^3 - 5b^2 - 60b$
69. $-2x^3 + 6x^2 - 8x$
70. $-4y^3 + 20y^2 - 32y$
71. $2m^4 - 10m^3 - 28m^2$
72. $6n^4 - 18n^3 + 12n^2$
73. $-3x^4 + 3x^3 + 6x^2$
74. $-5y^4 + 25y^3 - 30y^2$

Exercises 75–78: Factor the trinomial. (Hint: Write the expression in standard form.)

75. $5 + 6x + x^2$
76. $8 + 6x + x^2$
77. $3 - 4x + x^2$
78. $10 - 7x + x^2$

Exercises 79–82: Factor the trinomial. (Hint: Write $(m - x)(n + x)$ and find m and n.)

79. $12 + 4x - x^2$
80. $28 + 3x - x^2$
81. $32 - 4x - x^2$
82. $40 - 3x - x^2$

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83. **Thinking Generally** Factor the trinomial expression $x^2 + (k + 1)x + k$.

84. **Thinking Generally** Factor the trinomial expression $x^2 + (k - 2)x - 2k$.

GEOMETRY

85. A square has an area of $x^2 + 2x + 1$. Find the length of a side. Make a sketch of the square.

86. A square has an area of $x^2 + 6x + 9$. Find the length of a side. Make a sketch of the square.

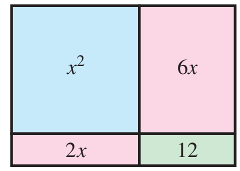
87. A rectangle has an area of $x^2 + 3x + 2$. Find one possibility for its width and length. Make a sketch of the rectangle.

88. A rectangle has an area of $x^2 + 9x + 8$. Find one possibility for its width and length. Make a sketch of the rectangle.

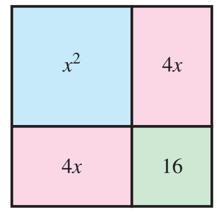
89. A cube has a surface area of $6x^2 + 12x + 6$. Find the length of a side. (*Hint:* First factor out the GCF.)

90. A cube has a surface area of $6x^2 + 36x + 54$. Find the length of a side.

91. Write a polynomial in factored form that represents the total area of the figure.



92. Write a polynomial in factored form that represents the total area of the figure.



WRITING ABOUT MATHEMATICS

93. Explain how to determine whether a trinomial has been factored correctly. Give an example.

94. Factoring $x^2 + bx + c$ involves finding two integers m and n such that $mn = c$ and $m + n = b$. Is it better first to determine values of m and n so that the product is c or the sum is b? Explain your reasoning.

SECTIONS 6.1 and 6.2 Checking Basic Concepts

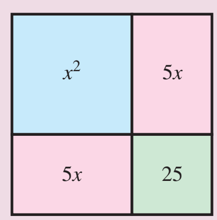
1. What is the greatest common factor for the expression $8x^3 - 12x^2 + 24x$?

2. Factor $12z^3 - 18z^2$.

3. Factor each expression completely.
(a) $6y(y - 2) + 5(y - 2)$
(b) $2x^3 + x^2 + 10x + 5$
(c) $4z^3 - 12z^2 + 4z - 12$

4. Factor each trinomial completely, if possible.
(a) $x^2 + 6x + 8$ **(b)** $x^2 - x - 42$
(c) $a^2 + 3a - 5$ **(d)** $4a^3 + 20a^2 + 24a$

5. Write a polynomial in factored form that represents the total area of the figure.



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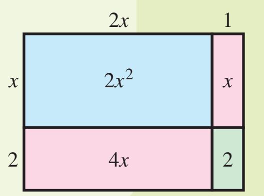
6.3 Factoring Trinomials II ($ax^2 + bx + c$)

Factoring Trinomials by Grouping • Factoring with FOIL in Reverse

A LOOK INTO MATH ▸



$2x^2 + 5x + 2 = (x + 2)(2x + 1)$

Figure 6.5

The sum of the areas of the four small rectangles shown in Figure 6.5 is $2x^2 + 5x + 2$. Note that the length of the large rectangle is $2x + 1$ and that its width is $x + 2$. One way to determine these dimensions is to factor $2x^2 + 5x + 2$. However, this trinomial has a leading coefficient of 2. Thus far we have factored only trinomials that have a leading coefficient of 1. In this section we discuss two methods used to factor trinomials in the form $ax^2 + bx + c$, where $a \neq 1$. In Example 1(a), we demonstrate how to factor the trinomial $2x^2 + 5x + 2$ as $(x + 2)(2x + 1)$.

Factoring Trinomials by Grouping

To factor the polynomial given by $x^2 + 6x + 5$ we find two numbers, m and n, such that $mn = 5$ and $m + n = 6$. For this trinomial we let $m = 1$ and $n = 5$, which gives

$$x^2 + 6x + 5 = (x + 1)(x + 5).$$

To factor the polynomial $2x^2 + 7x + 3$, which has a leading coefficient of 2, we must find two numbers m and n such that $mn = 2 \cdot 3 = 6$ and $m + n = 7$. One solution is $m = 1$ and $n = 6$. Using grouping, we can now factor this trinomial by writing $7x$ as the sum $1x + 6x$.

$$\begin{aligned} 2x^2 + 7x + 3 &= 2x^2 + \overbrace{x + 6x}^{7x} + 3 && \text{Write } 7x \text{ as } x + 6x. \\ &= (2x^2 + x) + (6x + 3) && \text{Associative property} \\ &= x(2x + 1) + 3(2x + 1) && \text{Factor out common factors.} \\ &= (x + 3)(2x + 1) && \text{Factor out } (2x + 1). \end{aligned}$$

This technique of factoring trinomials by grouping is summarized as follows.

FACTORING $ax^2 + bx + c$ BY GROUPING

To factor $ax^2 + bx + c$ perform the following steps. (Assume that a, b, and c are integers and have no common factors.)

1. Find two numbers, m and n, such that $mn = ac$ and $m + n = b$.
2. Write the trinomial as $ax^2 + mx + nx + c$.
3. Use grouping to factor this expression into two binomials.

STUDY TIP

Be sure that you understand that the values of m and n are used in different ways, depending on whether or not the leading coefficient is 1.

MAKING CONNECTIONS

The Values of m and n

The values of m and n found when factoring $ax^2 + bx + c$, where $a \neq 1$, are not used in the same way as the values of m and n found when factoring $x^2 + bx + c$.

Factoring $x^2 + bx + c$	***Factoring*** $ax^2 + bx + c$, ***where*** $a \neq 1$
$mn = c$ and $m + n = b$	$mn = ac$ and $m + n = b$
$x^2 + bx + c = (x + m)(x + n)$	$ax^2 + bx + c = ax^2 + mx + nx + c$
	Factor the resulting four-term polynomial by grouping.

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EXAMPLE 1

Factoring $ax^2 + bx + c$ by grouping

Factor each trinomial.

(a) $2x^2 + 5x + 2$ **(b)** $3z^2 + z - 2$ **(c)** $10t^2 - 11t + 3$

Solution

(a) To factor $2x^2 + 5x + 2$ we need to find m and n that satisfy $mn = 2 \cdot 2 = 4$ and $m + n = 5$. Table 6.8 shows that two such numbers are $m = 1$ and $n = 4$.

TABLE 6.8 Factor Pairs for 4

Factors	1, 4	2, 2
Sum	5	4

$$\begin{aligned} 2x^2 + 5x + 2 &= 2x^2 + \overbrace{x + 4x}^{5x} + 2 && \text{Write } 5x \text{ as } x + 4x. \\ &= (2x^2 + x) + (4x + 2) && \text{Associative property} \\ &= x(2x + 1) + 2(2x + 1) && \text{Factor out common factors.} \\ &= (x + 2)(2x + 1) && \text{Factor out } (2x + 1). \end{aligned}$$

(b) To factor $3z^2 + 1z - 2$ we need to find m and n that satisfy $mn = 3 \cdot (-2) = -6$ and $m + n = 1$. Two such numbers are $m = 3$ and $n = -2$.

$$\begin{aligned} 3z^2 + z - 2 &= 3z^2 + 3z - 2z - 2 && \text{Write } z \text{ as } 3z - 2z. \\ &= (3z^2 + 3z) + (-2z - 2) && \text{Associative property} \\ &= 3z(z + 1) - 2(z + 1) && \text{Factor out common factors.} \\ &= (3z - 2)(z + 1) && \text{Factor out } (z + 1). \end{aligned}$$

(c) To factor $10t^2 - 11t + 3$ we need to find m and n that satisfy $mn = 10 \cdot 3 = 30$ and $m + n = -11$. Two such numbers are $m = -5$ and $n = -6$.

$$\begin{aligned} 10t^2 - 11t + 3 &= 10t^2 - 5t - 6t + 3 && \text{Write } -11t \text{ as } -5t - 6t. \\ &= (10t^2 - 5t) + (-6t + 3) && \text{Associative property} \\ &= 5t(2t - 1) - 3(2t - 1) && \text{Factor out common factors.} \\ &= (5t - 3)(2t - 1) && \text{Factor out } (2t - 1). \end{aligned}$$

Now Try Exercises 15, 25, 37

MAKING CONNECTIONS

Different Ways to Factor by Grouping

In Example 1(c) we could have written $-11t$ as $-6t - 5t$, rather than $-5t - 6t$. Then the factoring could have been written as

$$\begin{aligned} 10t^2 - 11t + 3 &= 10t^2 - 6t - 5t + 3 \\ &= (10t^2 - 6t) + (-5t + 3) \\ &= 2t(5t - 3) - 1(5t - 3) \\ &= (2t - 1)(5t - 3), \end{aligned}$$

which gives the same result.

In the previous section we showed that some trinomials of the form $x^2 + bx + c$ are prime and cannot be factored into the product of two binomials with integer coefficients. The next example illustrates that some trinomials of the form $ax^2 + bx + c$, with $a \neq 1$, may also be prime.

EXAMPLE 2

Discovering that a trinomial is prime

Factor each trinomial.

(a) $3x^2 + 5x + 4$ **(b)** $2x^2 - 8x - 3$

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Solution

(a) To factor $3x^2 + 5x + 4$, we need to find integers m and n such that $mn = 3 \cdot 4 = 12$ and $m + n = 5$. Because the middle term is positive, we consider only positive factors of 12. Table 6.9 reveals that no such integers exist.

TABLE 6.9 Factor Pairs for 12

Factors	1, 12	2, 6	3, 4	
Sum	13	8	7	← No sum equals 5.

The trinomial $3x^2 + 5x + 4$ is prime.

(b) To factor $2x^2 - 8x - 3$, we must find integers m and n such that $mn = 2 \cdot (-3) = -6$ and $m + n = -8$. Table 6.10 reveals that no such integers exist.

TABLE 6.10 Factor Pairs for −6

Factors	−1, 6	−2, 3	2, −3	1, −6	
Sum	5	1	−1	−5	← No sum equals −8.

The trinomial $2x^2 - 8x - 3$ is prime.

Now Try Exercises 17, 27

Although some trinomials may look as though they can be factored using the process discussed in Example 1, it is important to remember to first factor out the greatest common factor whenever possible. In the next example we factor out the GCF before factoring the trinomial further.

EXAMPLE 3 Factoring out the GCF before factoring further

Factor each trinomial completely.

(a) $15x^2 - 50x - 40$ **(b)** $4x^3 - 22x^2 + 30x$

Solution

(a) Because the GCF of $15x^2 - 50x - 40$ is 5, factor out 5 before factoring the remaining trinomial.

$$15x^2 - 50x - 40 = 5(3x^2 - 10x - 8)$$

To factor $3x^2 - 10x - 8$, we need numbers m and n such that $mn = 3 \cdot (-8) = -24$ and $m + n = -10$. Two such numbers are -12 and 2.

$$\begin{aligned} 15x^2 - 50x - 40 &= 5(3x^2 - 12x + 2x - 8) && \text{Write } -10x \text{ as } -12x + 2x. \\ &= 5\big((3x^2 - 12x) + (2x - 8)\big) && \text{Associative property} \\ &= 5\big(3x(x - 4) + 2(x - 4)\big) && \text{Factor out common factors.} \\ &= 5(3x + 2)(x - 4) && \text{Factor out } (x - 4). \end{aligned}$$

(b) Because the GCF of $4x^3 - 22x^2 + 30x$ is $2x$, factor out $2x$ before factoring the remaining trinomial.

$$4x^3 - 22x^2 + 30x = 2x(2x^2 - 11x + 15)$$

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To factor the trinomial $2x^2 - 11x + 15$, we need to find numbers m and n such that $mn = 2 \cdot 15 = 30$ and $m + n = -11$. Two such numbers are -6 and -5.

$$\begin{aligned} 4x^3 - 22x^2 + 30x &= 2x(2x^2 - 6x - 5x + 15) && \text{Write } -11x \text{ as } -6x - 5x. \\ &= 2x\big((2x^2 - 6x) + (-5x + 15)\big) && \text{Associative property} \\ &= 2x\big(2x(x - 3) - 5(x - 3)\big) && \text{Factor out common factors.} \\ &= 2x(2x - 5)(x - 3) && \text{Factor out } (x - 3). \end{aligned}$$

Now Try Exercises 55, 59

Factoring with FOIL in Reverse

Rather than factoring a trinomial by grouping, we can use FOIL in reverse to determine its binomial factors. For example, the factors of $2x^2 + 5x + 2$ are two binomials:

$$2x^2 + 5x + 2 = (\_\_\_ + \_\_\_)(\_\_\_ + \_\_\_),$$

where the expressions to be placed in the four blanks are yet to be found. By the FOIL method, we know that the product of the first terms of these two binomials is $2x^2$. Because $2x^2 = 2x \cdot x$, we can write

$$2x^2 + 5x + 2 = (\underline{2x} + \_\_\_)(\underline{x} + \_\_\_).$$

By FOIL, the product of the last terms in these binomials must be 2. Because $2 = 1 \cdot 2$, we could put 1 and 2 in the blanks. However, we must be sure to place them correctly so that the product of the outside terms plus the product of the inside terms is $5x$.

$$(2x + 1)(x + 2) = 2x^2 + 5x + 2$$

$$1x + 4x = 5x \checkmark \leftarrow \text{Middle term checks.}$$

If we had interchanged the 1 and 2, we would have obtained an incorrect result.

$$(2x + 2)(x + 1) = 2x^2 + 4x + 2$$

$$2x + 2x = 4x \text{ ✗} \leftarrow \text{Middle term is } \textit{not } 5x.$$

MAKING CONNECTIONS

The Signs in the Binomial Factors

Let a, b, and c represent positive integers. If a trinomial of the form $ax^2 + bx + c$ can be factored, the signs in the binomial factors can be summarized as follows.

Form of the Trinomial	*Signs in the Binomial Factors*
$ax^2 + bx + c$	$(+)(+)$
$ax^2 - bx + c$	$(-)(-)$
$ax^2 + bx - c$	$(-)(+)$
$ax^2 - bx - c$	$(-)(+)$

In the next example we factor expressions of the form $ax^2 + bx + c$, where $a \neq 1$. We use FOIL in reverse and *trial and error* to find the correct factors.

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EXAMPLE 4 Factoring the form $ax^2 + bx + c$

Factor each trinomial.

(a) $3x^2 + 5x + 2$ **(b)** $6x^2 + 7x - 3$ **(c)** $6 + 4y^2 - 11y$

Solution

(a) To factor $3x^2 + 5x + 2$, we start by finding factors of $3x^2$, which include $\mathbf{3x}$ and $\mathbf{x}$, so we write

$$3x^2 + 5x + 2 = (3x + \_\_\_)(x + \_\_\_).$$

The factors of the last term 2 are **1** and **2**. (Because the middle term is positive, we do not consider -1 and -2.) If we place values of 1 and 2 in the binomial as follows, the middle term becomes $7x$ rather than $5x$.

$$(3x + 1)\ (x + 2) = 3x^2 + 7x + 2$$

$1x$
$+6x$
$7x$ ✗ ⟵ Middle term is *not* $5x$.

By reversing the positions of 1 and 2, we obtain a correct factorization.

$$(3x + 2)\ (x + 1) = 3x^2 + 5x + 2$$

$2x$
$+3x$
$5x$ ✓ ⟵ Middle term checks.

(b) To factor $6x^2 + 7x - 3$, we start by finding factors of $6x^2$, which include $2x$ and $3x$ or $6x$ and x. Factors of the last term, -3, include -1 and 3 or 1 and -3. The following two factorizations give incorrect results.

$$(2x + 1)\ (3x - 3) = 6x^2 - 3x - 3$$

$3x$
$-6x$
$-3x$ ✗ ⟵ Middle term is *not* $7x$.

$$(6x + 1)\ (x - 3) = 6x^2 - 17x - 3$$

$1x$
$-18x$
$-17x$ ✗ ⟵ Middle term is *not* $7x$.

To obtain a middle term of $7x$ we use the following arrangement.

$$(3x - 1)\ (2x + 3) = 6x^2 + 7x - 3$$

$-2x$
$+9x$
$7x$ ✓ ⟵ Middle term checks.

To find the correct factorization we often need to try more than one arrangement.

(c) To factor $6 + 4y^2 - 11y$, we start by writing the trinomial in standard form as $4y^2 - 11y + 6$. Then we find possible factors of the first term, $4y^2$, which include $2y$ and $2y$ or $4y$ and y. Factors of the last term, 6, include either -1 and -6 or -2 and -3. (Because the middle term is negative, we do not use the positive factors of 1 and 6 or 2 and 3.) To obtain a middle term of $-11y$ we use the following arrangement.

$$(4y - 3)\ (y - 2) = 4y^2 - 11y + 6$$

$-3y$
$-8y$
$-11y$ ✓ ⟵ Middle term checks.

Now Try Exercises 19, 33, 69

READING CHECK

- What is a good first step when factoring a trinomial with a negative leading coefficient?

In the next example we demonstrate how to factor trinomials having a negative leading coefficient. This task is sometimes accomplished by first factoring out -1.

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EXAMPLE 5 **Factoring trinomials having a negative leading coefficient**

Factor each trinomial.

(a) $-6x^2 + 17x - 5$ **(b)** $1 - x - 2x^2$

Solution

(a) One way to factor $-6x^2 + 17x - 5$ is to start by factoring out -1. Then we can apply FOIL in reverse.

$$\begin{aligned} -6x^2 + 17x - 5 &= -1\underbrace{(6x^2 - 17x + 5)}_{\text{Opposite of } -6x^2 + 17x - 5} && \text{Factor out } -1. \\ &= -1(3x - 1)(2x - 5) && \text{Factor the trinomial.} \\ &= -(3x - 1)(2x - 5) && \text{Rewrite.} \end{aligned}$$

(b) To factor the trinomial $1 - x - 2x^2$, write it in standard form, factor out -1, and then apply FOIL in reverse.

$$\begin{aligned} 1 - x - 2x^2 &= -2x^2 - x + 1 && \text{Standard form} \\ &= -1(2x^2 + x - 1) && \text{Factor out } -1. \\ &= -(2x - 1)(x + 1) && \text{Factor the trinomial.} \end{aligned}$$

Now Try Exercises 71, 75

6.3 Putting It All Together

CONCEPT	EXPLANATION	EXAMPLES
Factoring Trinomials by Grouping	To factor $ax^2 + bx + c$, find two numbers, m and n, such that $mn = ac$ and $m + n = b$. Then write $ax^2 + mx + nx + c.$ Use grouping to factor this expression into two binomials.	For $3x^2 + 10x + 8$ let $m = 6$ and $n = 4$ because $mn = 24$ and $m + n = 10$. $3x^2 + 10x + 8 = 3x^2 + 6x + 4x + 8$ $= (3x^2 + 6x) + (4x + 8)$ $= 3x(x + 2) + 4(x + 2)$ $= (3x + 4)(x + 2)$
Factoring Trinomials by Using FOIL in Reverse	To factor $ax^2 + bx + c$, find factors of ax^2 and of c. Choose and arrange these factors in two binomials so that the middle term is bx.	For $3x^2 + 10x + 8$ the factors of $3x^2$ are $3x$ and x. The positive factors of 8 are either 2 and 4, or 1 and 8. A middle term of $10x$ can be obtained as follows. $(3x + 4)(x + 2) = 3x^2 + 10x + 8$ $4x$ $+6x$ $10x$ ✓ ← Middle term checks.

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CONCEPT	EXPLANATION	EXAMPLES
Choosing Signs When Factoring $ax^2 + bx + c$	For $ax^2 + bx + c$ with $a > 0$, **1.** If $c > 0$ and $b > 0$, use $(\underline{\quad} + \underline{\quad})(\underline{\quad} + \underline{\quad})$. **2.** If $c > 0$ and $b < 0$, use $(\underline{\quad} - \underline{\quad})(\underline{\quad} - \underline{\quad})$. **3.** If $c < 0$, use $(\underline{\quad} - \underline{\quad})(\underline{\quad} + \underline{\quad})$ or $(\underline{\quad} + \underline{\quad})(\underline{\quad} - \underline{\quad})$.	**1.** $2x^2 + 7x + 3 = (2x + 1)(x + 3)$ $(c = 3, b = 7)$ **2.** $2x^2 - 7x + 3 = (2x - 1)(x - 3)$ $(c = 3, b = -7)$ **3.** $2x^2 + 5x - 3 = (2x - 1)(x + 3)$ $(c = -3, b = 5)$ $2x^2 - 5x - 3 = (2x + 1)(x - 3)$ $(c = -3, b = -5)$

6.3 Exercises

CONCEPTS AND VOCABULARY

1. To factor the polynomial $ax^2 + bx + c$ by grouping, you first find two numbers, m and n, such that $mn = \underline{\qquad}$ and $m + n = \underline{\qquad}$.

2. To factor the polynomial $ax^2 + bx + c$ with FOIL in reverse, you first find possible factors for _____ and for _____.

Exercises 3–6: Insert the symbol + or − in each binomial factor to make the equation true.

3. $3x^2 + 5x + 2 = (3x\underline{\quad}2)(x\underline{\quad}1)$.
4. $3x^2 - x - 2 = (3x\underline{\quad}2)(x\underline{\quad}1)$.
5. $3x^2 - 5x + 2 = (3x\underline{\quad}2)(x\underline{\quad}1)$.
6. $3x^2 + x - 2 = (3x\underline{\quad}2)(x\underline{\quad}1)$.

Exercises 7–14: Fill in the blank in each binomial factor to make the equation true.

7. $4x^2 + 11x + 6 = (4x + \underline{\quad})(\underline{\quad} + 2)$.
8. $4x^2 - 5x - 6 = (\underline{\quad} - 2)(\underline{\quad} + 3)$.
9. $4x^2 + 4x - 3 = (2x - \underline{\quad})(2x + \underline{\quad})$.
10. $4x^2 - 8x + 3 = (2x - \underline{\quad})(\underline{\quad} - 3)$.
11. $6x^2 + 11x - 7 = (\underline{\quad} + 7)(2x - \underline{\quad})$.
12. $6x^2 - 31x + 28 = (\underline{\quad} - 7)(\underline{\quad} - 4)$.
13. $6x^2 - x - 15 = (3x - \underline{\quad})(2x + \underline{\quad})$.
14. $6x^2 - 53x + 40 = (\underline{\quad} - 5)(x - \underline{\quad})$.

FACTORING TRINOMIALS

Exercises 15–54: Factor the trinomial. If the trinomial cannot be factored, write "prime."

15. $2x^2 + 7x + 3$
16. $2x^2 + 3x + 1$
17. $3y^2 + 2y + 4$
18. $2y^2 + 5y + 1$
19. $3x^2 + 4x + 1$
20. $3x^2 + 10x + 3$
21. $6x^2 + 11x + 3$
22. $6x^2 + 17x + 5$
23. $5x^2 - 11x + 2$
24. $7x^2 - 8x + 1$
25. $2y^2 - 7y + 5$
26. $2y^2 - 11y + 12$
27. $3m^2 - 11m - 6$
28. $5m^2 - 7m - 2$
29. $7z^2 - 37z + 10$
30. $3z^2 - 11z + 6$
31. $3t^2 - 7t - 6$
32. $8t^2 - 6t - 9$
33. $15r^2 + r - 6$
34. $12r^2 + r - 6$
35. $24m^2 - 23m - 12$
36. $24m^2 + 29m - 4$
37. $25x^2 + 5x - 2$
38. $30x^2 + 7x - 2$
39. $6x^2 + 11x - 2$
40. $12x^2 + 28x - 5$
41. $15y^2 - 7y + 2$
42. $14y^2 - 5y + 1$
43. $21n^2 + 4n - 1$
44. $21n^2 + 10n + 1$
45. $14y^2 + 23y + 3$
46. $28y^2 + 25y + 3$
47. $28z^2 - 25z + 3$
48. $15z^2 - 19z + 6$
49. $30x^2 - 29x + 6$
50. $50x^2 - 55x + 12$

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51. $20a^2 + 18a - 5$

52. $40a^2 + 21a - 2$

53. $18t^2 + 23t - 6$

54. $33t^2 + 7t - 10$

Exercises 55–64: Factor the trinomial completely.

55. $12a^2 + 12a - 9$

56. $21b^2 - 14b - 56$

57. $12y^3 - 11y^2 + 2y$

58. $10z^3 + 19z^2 + 6z$

59. $24x^3 - 30x^2 + 9x$

60. $8y^3 - 16y^2 + 6y$

61. $8x^4 - 6x^3 + 2x^2$

62. $10y^3 + 15y^2 - 5y$

63. $28x^4 + 56x^3 + 21x^2$

64. $20y^4 + 42y^3 - 20y^2$

65. **Thinking Generally** Factor the trinomial expression $3x^2 + (3k + 1)x + k$.

66. **Thinking Generally** Factor the trinomial expression $3x^2 + (3k - 2)x - 2k$.

Exercises 67–76: Factor the trinomial.

67. $2 + 15x + 7x^2$

68. $3 + 16x + 5x^2$

69. $2 - 5x + 2x^2$

70. $5 - 6x + x^2$

71. $3 - 2x - 8x^2$

72. $5 - 3x - 2x^2$

73. $-2x^2 - 7x + 15$

74. $-5x^2 - 19x + 4$

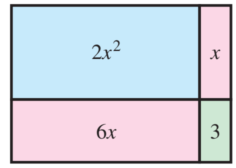
75. $-5x^2 + 14x + 3$

76. $-6x^2 + 17x + 14$

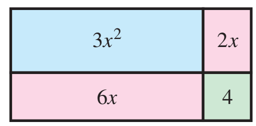
77. A rectangle has an area of $6x^2 + 7x + 2$. Find possible dimensions for this rectangle. Make a sketch of the rectangle.

78. A rectangle has an area of $2x^2 + 5x + 3$. Find possible dimensions for the rectangle. Make a sketch of the rectangle.

79. Write a polynomial in factored form that represents the total area of the figure.



80. Write a polynomial in factored form that represents the total area of the figure.



WRITING ABOUT MATHEMATICS

81. Explain how the sign of the third term in the trinomial $ax^2 + bx + c$ affects how it is factored.

82. Explain the steps to be used to factor $ax^2 + bx + c$ by grouping.

Group Activity Working with Real Data

Directions: Form a group of 2 to 4 people. Select someone to record the group's responses for this activity. All members of the group should work cooperatively to answer the questions. If your instructor asks for your results, each member of the group should be prepared to respond.

AIDS Cases From 1993 to 2009 the cumulative number N of AIDS cases in thousands can be approximated by $N = -2x^2 + 76x + 430$, where $x = 0$ corresponds to the year 1993.

Year	1993	1997	2001	2005	2009
Cases	422	677	844	984	1109

Source: U.S. Department of Health and Human Services.

(a) Use the equation to find N for each year in the table.

(b) Discuss how well this equation approximates the data.

(c) Rewrite the equation with the right side completely factored.

(d) Use your equation from part (c) to find N for each year in the table. Do your answers agree with those found in part (a)?

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6.4 Special Types of Factoring

Difference of Two Squares • Perfect Square Trinomials • Sum and Difference of Two Cubes

A LOOK INTO MATH ▶

When children are first learning to read, they often spend a great deal of time "sounding out" the words that they encounter. In time, some words become easily recognizable as *sight words* that can be read more quickly. In mathematics, we can learn to recognize certain types of polynomials that can be factored more quickly using special factoring techniques.

NEW VOCABULARY

☐ Perfect square trinomial

STUDY TIP

The special types of factoring discussed in this section are helpful for factoring some (but not all) types of polynomials. Don't forget to review factoring techniques that were discussed earlier in this chapter.

Difference of Two Squares

In Section 5.4 we showed that $(a - b)(a + b) = a^2 - b^2$. If we rewrite this equation as

$$a^2 - b^2 = (a - b)(a + b),$$

then we can use it to factor a difference of two squares. For example, to factor the expression $x^2 - 25$, we first write it in the form $a^2 - b^2$, where $a = x$ and $b = 5$. Doing so results in expressing $x^2 - 25$ as $x^2 - 5^2$, and the equation

$$a^2 - b^2 = (a - b)(a + b)$$

becomes

$$x^2 - 5^2 = (x - 5)(x + 5).$$

This discussion suggests the following rule for factoring a difference of two squares.

DIFFERENCE OF TWO SQUARES

For any real numbers a and b,

$$a^2 - b^2 = (a - b)(a + b).$$

In the next example we apply this method to other expressions.

EXAMPLE 1 **Factoring the difference of two squares**

Factor each difference of two squares.
(a) $x^2 - 36$ **(b)** $4x^2 - 9$ **(c)** $100 - 16t^2$ **(d)** $49y^2 - 64z^2$

Solution

(a) Write $x^2 - 36$ as $x^2 - 6^2$ and then substitute x for a and 6 for b. The equation

$$a^2 - b^2 = (a - b)(a + b)$$

becomes

$$x^2 - 6^2 = (x - 6)(x + 6).$$

(b) The expression $4x^2 - 9$ can be written as $(2x)^2 - 3^2$. Thus

$$4x^2 - 9 = (2x - 3)(2x + 3).$$

(c) The expression $100 - 16t^2$ can be written as $(10)^2 - (4t)^2$. Thus

$$100 - 16t^2 = (10 - 4t)(10 + 4t).$$

(d) The expression $49y^2 - 64z^2$ can be written as $(7y)^2 - (8z)^2$. Thus

$$49y^2 - 64z^2 = (7y - 8z)(7y + 8z).$$

Now Try Exercises 17, 19, 25, 29

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READING CHECK

- Is it possible to factor a *sum* of two squares by using real numbers?

MAKING CONNECTIONS

Sum of Squares versus Difference of Squares

The sum of two squares, $a^2 + b^2$, cannot be factored by using real numbers. However, the difference of two squares, $a^2 - b^2$, can be factored. For example, $x^2 + 4$ cannot be factored, but $x^2 - 4$ can be factored as $(x - 2)(x + 2)$.

Perfect Square Trinomials

In Section 5.4 we also showed how to compute $(a + b)^2$ and $(a - b)^2$ as

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \text{and}$$
$$(a - b)^2 = a^2 - 2ab + b^2.$$

The expressions $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ are called **perfect square trinomials.** If we can recognize a perfect square trinomial, we can use the following formulas to factor it.

PERFECT SQUARE TRINOMIALS

For any real numbers a and b,

$$a^2 + 2ab + b^2 = (a + b)^2 \quad \text{and}$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

READING CHECK

- How can you recognize a perfect square trinomial?

When factoring a trinomial as a perfect square trinomial, we must first verify that the middle term is correct. For example, to factor $x^2 + 14x + 49$ we start by rewriting the expression as $x^2 + 14x + 7^2$. In order to factor this trinomial as a perfect square trinomial, the middle term $14x$ must be equal to twice the product of x and 7.

$$x^2 + 14x + 7^2$$
$$2(x)(7) \ ✓$$ The middle term checks.

When $a = x$ and $b = 7$, the equation $a^2 + 2ab + b^2 = (a + b)^2$ allows us to factor the given polynomial as $x^2 + 14x + 49 = (x + 7)^2$.

EXAMPLE 2 **Factoring perfect square trinomials**

If possible, factor each trinomial as a perfect square trinomial.
(a) $x^2 + 10x + 25$ **(b)** $4x^2 - 4x + 1$ **(c)** $9z^2 + 18z + 4$ **(d)** $x^2 - 4xy + 4y^2$

Solution

(a) Start by writing $x^2 + 10x + 25$ as $x^2 + 10x + 5^2$ and then check the middle term.

$$x^2 + 10x + 5^2$$
$$2(x)(5) \ ✓$$ The middle term checks.

When $a = x$ and $b = 5$, the equation $a^2 + 2ab + b^2 = (a + b)^2$ allows us to factor the given polynomial as $x^2 + 10x + 25 = (x + 5)^2$.

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(b) The polynomial $4x^2 - 4x + 1$ can be written as $(2x)^2 - 4x + 1^2$.

$$(2x)^2 - 4x + 1^2$$
$$2(2x)(1) \checkmark$$ The middle term checks.

When $a = 2x$ and $b = 1$, the equation $a^2 - 2ab + b^2 = (a - b)^2$ allows us to factor the given polynomial as $4x^2 - 4x + 1 = (2x - 1)^2$.

(c) Write $9z^2 + 18z + 4$ as $(3z)^2 + 18z + 2^2$ and then check the middle term.

$$(3z)^2 + 18z + 2^2$$
$$2(3z)(2) \times$$ The middle term is $12z$ and does not equal $18z$.

The expression $9z^2 + 18z + 4$ cannot be factored as a perfect square trinomial.

(d) The polynomial $x^2 - 4xy + 4y^2$ can be written as $x^2 - 4xy + (2y)^2$.

$$x^2 - 4xy + (2y)^2$$
$$2(x)(2y) \checkmark$$ The middle term checks.

When $a = x$ and $b = 2y$, the equation $a^2 - 2ab + b^2 = (a - b)^2$ allows us to factor the given polynomial as $x^2 - 4xy + 4y^2 = (x - 2y)^2$.

Now Try Exercises 37, 41, 43, 51

MAKING CONNECTIONS

Special Factoring and General Techniques

If you do not recognize a polynomial as the difference of two squares or a perfect square trinomial, you can still factor the polynomial by using the methods discussed in earlier sections.

Sum and Difference of Two Cubes

The sum or difference of two cubes may be factored—a result of the two equations

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3 \quad \text{and}$$
$$(a - b)(a^2 + ab + b^2) = a^3 - b^3.$$

These equations can be verified by multiplying the left side to obtain the right side. For example, multiplying the polynomials on the left side of the first equation results in

$$\begin{aligned}(a + b)(a^2 - ab + b^2) &= a \cdot a^2 - a \cdot ab + a \cdot b^2 + b \cdot a^2 - b \cdot ab + b \cdot b^2 \\ &= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 \\ &= a^3 + b^3.\end{aligned}$$

SUM AND DIFFERENCE OF TWO CUBES

For any real numbers a and b,

Opposite Signs

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad \text{and}$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Opposite Signs

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Any binomial whose terms can be expressed as cubes can be factored as a sum or difference of cubes. We demonstrate this method in the next example.

EXAMPLE 3 **Factoring the sum and difference of two cubes**

Factor each polynomial.
(a) $z^3 + 8$ **(b)** $x^3 - 27$ **(c)** $8x^3 - 1$

Solution

(a) To factor $z^3 + 8$ we let $a^3 = z^3$ and $b^3 = 2^3$ so that $a = z$ and $b = 2$. Then

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

becomes

$$\begin{aligned} z^3 + 2^3 &= (z + 2)(z^2 - z \cdot 2 + 2^2) \\ &= (z + 2)(z^2 - 2z + 4). \end{aligned}$$

(b) To factor $x^3 - 27$ we let $a^3 = x^3$ and $b^3 = 3^3$ so that $a = x$ and $b = 3$. Then

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

becomes

$$\begin{aligned} x^3 - 3^3 &= (x - 3)(x^2 + x \cdot 3 + 3^2) \\ &= (x - 3)(x^2 + 3x + 9). \end{aligned}$$

(c) To factor $8x^3 - 1$ we let $a^3 = (2x)^3$ and $b^3 = 1^3$ so that $a = 2x$ and $b = 1$. Be sure to write $2x$ in **parentheses** when substituting for a in the term a^2.

$$\begin{aligned} (2x)^3 - 1^3 &= (2x - 1)\left((2x)^2 + 2x \cdot 1 + 1^2\right) \\ &= (2x - 1)(4x^2 + 2x + 1) \end{aligned}$$

Now Try Exercises 57, 65

In this section we have discussed special methods for factoring polynomials that can be identified as the difference of two squares, perfect square trinomials, the sum of two cubes, or the difference of two cubes. The next example demonstrates how to recognize and factor such polynomials.

EXAMPLE 4 **Recognizing polynomials that can be factored with special methods**

Factor each polynomial.
(a) $8x^3 + 27$ **(b)** $4y^2 - 20y + 25$ **(c)** $9z^2 - 64$

Solution

(a) Because this polynomial has only two terms, it cannot be factored as a perfect square trinomial. Since it is not a difference, it cannot be factored as the difference of two squares. We will try to factor the polynomial as the sum of two cubes. To factor $8x^3 + 27$, we note that $8x^3 = (2x)^3$ and $27 = 3^3$. Then

$$\begin{aligned} 8x^3 + 27 &= (2x)^3 + 3^3 \\ &= (2x + 3)\left((2x)^2 - 2x \cdot 3 + 3^2\right) \\ &= (2x + 3)(4x^2 - 6x + 9). \end{aligned}$$

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(b) Because this polynomial has three terms, we will try to factor it as a perfect square trinomial. Begin by writing $4y^2 - 20y + 25$ as $(2y)^2 - 20y + 5^2$ and then verify that the middle term checks.

$$(2y)^2 - 20y + 5^2$$
$$2(2y)(5) \checkmark \quad \text{The middle term checks.}$$

When $a = 2y$ and $b = 5$, the equation $a^2 - 2ab + b^2 = (a - b)^2$ allows us to factor the given polynomial as $4y^2 - 20y + 25 = (2y - 5)^2$.

(c) Because this polynomial is a difference of two terms that appear to be square terms, we will try to factor the polynomial as the difference of two squares. The expression $9z^2 - 64$ can be written as $(3z)^2 - 8^2$. Thus

$$9z^2 - 64 = (3z - 8)(3z + 8).$$

Now Try Exercises 31, 47, 67

When using the special factoring methods discussed in this section, it is important to remember to first factor out the greatest common factor whenever possible. In the next example we factor out the GCF before factoring further.

EXAMPLE 5 Factoring out the GCF before factoring further

Factor each polynomial completely.

(a) $27x^3 + 72x^2 + 48x$ **(b)** $18a^3 - 8ab^2$

Solution

(a) Because the GCF of $27x^3 + 72x^2 + 48x$ is $3x$, factor out $3x$ before factoring the remaining trinomial.

$$27x^3 + 72x^2 + 48x = 3x(9x^2 + 24x + 16)$$

The expression $9x^2 + 24x + 16$ is a perfect square trinomial that can be written as $(3x)^2 + 24x + 4^2$.

$$(3x)^2 + 24x + 4^2$$
$$2(3x)(4) \checkmark \quad \text{The middle term checks.}$$

Thus $9x^2 + 24x + 16 = (3x + 4)^2$. As a result, the given polynomial can be factored as

$$27x^3 + 72x^2 + 48x = 3x(3x + 4)^2.$$

(b) Because the GCF of $18a^3 - 8ab^2$ is $2a$, factor out $2a$ before factoring the remaining polynomial.

$$18a^3 - 8ab^2 = 2a(9a^2 - 4b^2)$$

The expression $9a^2 - 4b^2$ is the difference of two squares and can be written as $(3a)^2 - (2b)^2$. Thus

$$18a^3 - 8ab^2 = 2a(3a - 2b)(3a + 2b).$$

Now Try Exercises 71, 79

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6.4 Putting It All Together

FACTORING	EXPLANATION	EXAMPLES
Difference of Two Squares	$a^2 - b^2 = (a - b)(a + b)$ **NOTE:** The *sum* of two squares, $a^2 + b^2$, cannot be factored by using real numbers.	$y^2 - 49 = (y - 7)(y + 7)$ $81 - z^2 = (9 - z)(9 + z)$ $4r^2 - 25t^2 = (2r - 5t)(2r + 5t)$ $16a^2 + b^2$ cannot be factored.
Perfect Square Trinomial	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$ Be sure to verify that the given middle term equals $2ab$ before factoring.	$m^2 + 2m + 1 = (m + 1)^2$ $25y^2 - 30y + 9 = (5y - 3)^2$ $36r^2 + 12rt + t^2 = (6r + t)^2$ $x^2 + 5x + 4$ is *not* a perfect square trinomial because $2ab = 2 \cdot x \cdot 2 = 4x \neq 5x$.
Sum and Difference of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	$y^3 + 27 = (y + 3)(y^2 - y \cdot 3 + 3^2)$ $= (y + 3)(y^2 - 3y + 9)$ $27r^3 - 64t^3$ $= (3r - 4t)\left((3r)^2 + 3r \cdot 4t + (4t)^2\right)$ $= (3r - 4t)(9r^2 + 12rt + 16t^2)$

6.4 Exercises

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CONCEPTS AND VOCABULARY

1. $a^2 - b^2 =$ _____

2. If the expression $36x^2 - 49y^2$ is written in the form $a^2 - b^2$, then $a =$ _____ and $b =$ _____.

3. (True or False?) The expression $a^2 + b^2$ can be factored by using real numbers.

4. (True or False?) Using only integer coefficients, the expression $3x^2 - 5y^2$ can be factored as a difference of two squares.

5. $a^2 + 2ab + b^2 =$ _____

6. $a^2 - 2ab + b^2 =$ _____

7. $x^2 +$ _____ $+ 9$ is a perfect square trinomial.

8. $4r^2 -$ _____ $+ 25t^2$ is a perfect square trinomial.

9. $a^3 + b^3 =$ _____

10. $a^3 - b^3 =$ _____

11. If the expression $8x^3 + 27y^3$ is written in the form $a^3 + b^3$, then $a =$ _____ and $b =$ _____.

12. If the expression $x^3 - 1$ is written in the form $a^3 - b^3$, then $a =$ _____ and $b =$ _____.

13. $y^3 - 8 = (y$ __ $2)(y^2$ __ $2y + 4)$

14. $64z^3 + 27 = (4z$ __ $3)(16z^2$ __ $12z + 9)$

FACTORING THE DIFFERENCE OF TWO SQUARES

Exercises 15–32: Factor.

15. $x^2 - 1$

16. $x^2 - 16$

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17. $z^2 - 100$

18. $z^2 - 81$

19. $4y^2 - 1$

20. $9y^2 - 16$

21. $36z^2 - 25$

22. $49z^2 - 64$

23. $9 - x^2$

24. $25 - x^2$

25. $1 - 9y^2$

26. $49 - 16y^2$

27. $4a^2 - 9b^2$

28. $16a^2 - b^2$

29. $36m^2 - 25n^2$

30. $49m^2 - 100n^2$

31. $81r^2 - 49t^2$

32. $625r^2 - 121t^2$

FACTORING PERFECT SQUARE TRINOMIALS

Exercises 33–54: Factor as a perfect square trinomial whenever possible.

33. $x^2 + 8x + 16$

34. $x^2 + 4x + 4$

35. $z^2 + 12z + 25$

36. $z^2 - 18z + 36$

37. $x^2 - 6x + 9$

38. $x^2 - 10x + 25$

39. $9y^2 + 6y + 1$

40. $16y^2 + 8y + 1$

41. $4z^2 - 4z + 1$

42. $25z^2 - 12z + 1$

43. $9t^2 + 16t + 4$

44. $4t^2 + 12t + 9$

45. $9x^2 + 30x + 25$

46. $25x^2 + 60x + 36$

47. $4a^2 - 36a + 81$

48. $9a^2 - 60a + 100$

49. $x^2 + 2xy + y^2$

50. $x^2 - 6xy + 9y^2$

51. $r^2 - 10rt + 25t^2$

52. $15r^2 + 10rt + t^2$

53. $4y^2 - 10yz + 9z^2$

54. $25y^2 - 20yz + 4z^2$

FACTORING SUMS AND DIFFERENCES OF TWO CUBES

Exercises 55–68: Factor.

55. $z^3 + 1$

56. $z^3 + 8$

57. $x^3 + 64$

58. $x^3 + 125$

59. $y^3 - 8$

60. $y^3 - 27$

61. $n^3 - 1$

62. $n^3 - 64$

63. $8x^3 + 1$

64. $27x^3 - 1$

65. $m^3 - 64n^3$

66. $m^3 + 8n^3$

67. $8x^3 + 125y^3$

68. $27x^3 + 64y^3$

GENERAL FACTORING USING SPECIAL METHODS

Exercises 69–86: Factor the expression completely.

69. $4x^2 - 16$

70. $12x^2 - 60x + 75$

71. $2y^2 - 28y + 98$

72. $y^3 - 9y$

73. $5z^3 + 40$

74. $4z^3 + 36z^2 + 100z$

75. $x^3y - xy^3$

76. $8m^3 - 8$

77. $2m^3 - 10m^2 + 18m$

78. $2a^3b - 18ab^3$

79. $700x^4 - 63x^2y^2$

80. $135r^3 - 5t^3$

81. $16a^3 + 2b^3$

82. $192x^2y^2 - 3y^4$

83. $4b^4 + 24b^3 + 36b^2$

84. $2y^4 + 24y^3 + 72y^2$

85. $500r^3 - 32t^3$

86. $8r^3 - 64t^3$

GEOMETRY

87. A square has an area of $4x^2 + 12x + 9$. Find the length of a side. Make a sketch of the square.

88. A square has an area of $9x^2 + 30x + 25$. Find the length of a side. Make a sketch of the square.

WRITING ABOUT MATHEMATICS

89. Explain how factoring $x^3 + y^3$ is different from factoring $x^3 - y^3$.

90. Using the techniques discussed in this section, can you factor the expression $4x^2 + 9y^2$ into two binomials? Explain your reasoning.

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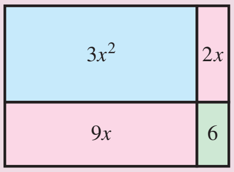
SECTIONS 6.3 and 6.4

Checking Basic Concepts

1. Factor each trinomial.
   (a) $2x^2 - 5x - 12$ **(b)** $6x^2 + 17x - 14$

2. Factor completely when possible.
   (a) $3y^2 + 4y - 2$ **(b)** $6y^3 - 10y^2 - 4y$

3. Write a polynomial in factored form that represents the total area of the figure.



4. Factor each polynomial.
   (a) $z^2 - 64$ **(b)** $9r^2 - 4t^2$

5. Factor each trinomial.
   (a) $x^2 + 12x + 36$ **(b)** $9a^2 - 12ab + 4b^2$

6. Factor.
   (a) $m^3 - 27$ **(b)** $125n^3 + 27$

7. Factor completely.
   (a) $16x^2 - 4$ **(b)** $3y^4 + 24y$

6.5 Summary of Factoring

Guidelines for Factoring Polynomials • Factoring Polynomials

A LOOK INTO MATH ▶

So far in this chapter we have discussed several useful techniques for factoring polynomials. But in most factoring problems, the specific method that should be used is not stated. Instead, we must look carefully at each factoring problem and decide which approach is best. In this section we discuss general guidelines that can be used to factor polynomials.

Guidelines for Factoring Polynomials

The following guidelines can be used to factor polynomials in general.

STUDY TIP

This section provides an opportunity to practice the material from several sections. In general, it is a good idea to regularly review topics that have been covered earlier.

FACTORING POLYNOMIALS

STEP 1: Factor out the greatest common factor (GCF), if possible.

STEP 2: **A.** If the polynomial has *four terms*, try factoring by grouping.

B. If the polynomial is a *binomial*, try one of the following.

1. $a^2 - b^2 = (a - b)(a + b)$ Difference of two squares
2. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ Difference of two cubes
3. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ Sum of two cubes

C. If the polynomial is a *trinomial*, check for a perfect square.

1. $a^2 + 2ab + b^2 = (a + b)^2$ Perfect square trinomial
2. $a^2 - 2ab + b^2 = (a - b)^2$ Perfect square trinomial

Otherwise, try to factor the trinomial by grouping or apply FOIL in reverse, as described in Sections 6.2 and 6.3.

STEP 3: Check to make sure that the polynomial is *completely* factored.

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READING CHECK

- In your own words, rewrite the general guidelines for factoring polynomials.

NOTE: Always perform Step 1 first. Factoring out the greatest common factor usually makes it easier to factor the resulting polynomial. After a polynomial has been factored, remember to perform Step 3 so that you are sure the given polynomial is completely factored.

Factoring Polynomials

In the first example, we apply Step 1 to a polynomial with a common factor.

EXAMPLE 1 **Factoring out a common factor**

Factor $5x^3 - 10x^2 + 15x$.

Solution

STEP 1: The greatest common factor is $5x$.

$$5x^3 - 10x^2 + 15x = 5x(x^2 - 2x + 3)$$

STEP 2C: The trinomial $x^2 - 2x + 3$ is prime and cannot be factored further.

STEP 3: The completely factored polynomial is $5x(x^2 - 2x + 3)$.

Now Try Exercise 11

When factoring polynomials completely, it is often necessary to apply more than one factoring technique. In several of the next examples we factor polynomials that require more than one method of factoring.

EXAMPLE 2 **Factoring a difference of squares**

Factor $3x^4 - 48x^2$.

Solution

STEP 1: The greatest common factor is $3x^2$.

$$3x^4 - 48x^2 = 3x^2(x^2 - 16)$$

STEP 2B: The binomial $x^2 - 16$ can be factored as a difference of two squares.

$$3x^2(x^2 - 16) = 3x^2(x - 4)(x + 4)$$

STEP 3: The completely factored polynomial is $3x^2(x - 4)(x + 4)$.

Now Try Exercise 39

MAKING CONNECTIONS

Factoring Polynomials

We can often determine how a polynomial should be factored by considering the number of terms in the polynomial. This is summarized as follows.

Type of Polynomial	*Factoring Technique*
4-term Polynomial	Grouping
Trinomial	Perfect square trinomial FOIL in reverse or grouping
Binomial	Difference of two squares Sum or difference of two cubes

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EXAMPLE 3 **Factoring a perfect square trinomial**

Factor $36y^3 - 24y^2 + 4y$.

Solution

STEP 1: The greatest common factor is $4y$.

$$36y^3 - 24y^2 + 4y = 4y(9y^2 - 6y + 1)$$

STEP 2C: We can factor $9y^2 - 6y + 1$ as a perfect square trinomial.

$$4y(9y^2 - 6y + 1) = 4y(3y - 1)(3y - 1)$$

STEP 3: The completely factored polynomial is $4y(3y - 1)^2$.

Now Try Exercise 41

EXAMPLE 4 **Factoring a sum of cubes**

Factor $27z^3 + 64$.

Solution

STEP 1: There are no common factors.

STEP 2B: The binomial $27z^3 + 64$ can be written as $(3z)^3 + 4^3$ and can be factored as a sum of two cubes.

$$\begin{aligned} 27z^3 + 64 &= (3z + 4)\left((3z)^2 - 3z \cdot 4 + 4^2\right) \\ &= (3z + 4)(9z^2 - 12z + 16) \end{aligned}$$

NOTE: The trinomial $9z^2 - 12z + 16$ is prime and cannot be factored further.

STEP 3: The completely factored polynomial is $(3z + 4)(9z^2 - 12z + 16)$.

Now Try Exercise 31

EXAMPLE 5 **Factoring a trinomial**

Factor $14x^4 + 7x^3 - 42x^2$.

Solution

STEP 1: The greatest common factor is $7x^2$.

$$14x^4 + 7x^3 - 42x^2 = 7x^2(2x^2 + x - 6)$$

STEP 2C: We can factor $2x^2 + x - 6$ using FOIL in reverse.

$$7x^2(2x^2 + x - 6) = 7x^2(2x - 3)(x + 2)$$

STEP 3: The completely factored polynomial is $7x^2(2x - 3)(x + 2)$.

Now Try Exercise 35

EXAMPLE 6 **Factoring by grouping**

Factor $15x^3 + 10x^2 - 60x - 40$.

Solution

STEP 1: The greatest common factor is 5.

$$15x^3 + 10x^2 - 60x - 40 = 5(3x^3 + 2x^2 - 12x - 8)$$

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STEP 2A: Because the resulting polynomial has four terms, we apply grouping.

$$5(3x^3 + 2x^2 - 12x - 8) = 5\big((3x^3 + 2x^2) + (-12x - 8)\big) \quad \text{Associative property}$$
$$= 5\big((x^2)(3x + 2) - 4(3x + 2)\big) \quad \text{Factor out common factors.}$$
$$= 5(x^2 - 4)(3x + 2) \quad \text{Factor out } (3x + 2).$$

STEP 2B: The binomial $x^2 - 4$ can now be factored as a difference of two squares.

$$5(x^2 - 4)(3x + 2) = 5(x - 2)(x + 2)(3x + 2)$$

STEP 3: The completely factored polynomial is $5(x - 2)(x + 2)(3x + 2)$.

Now Try Exercise 33

EXAMPLE 7 Factoring a polynomial having two variables

Factor $18x^3y - 8xy^3$.

Solution

STEP 1: The greatest common factor is $2xy$.

$$18x^3y - 8xy^3 = 2xy(9x^2 - 4y^2)$$

STEP 2B: The binomial $9x^2 - 4y^2$ can be written as $(3x)^2 - (2y)^2$ and can be factored as a difference of two squares.

$$9x^2 - 4y^2 = (3x - 2y)(3x + 2y)$$

STEP 3: The completely factored polynomial is $2xy(3x - 2y)(3x + 2y)$.

Now Try Exercise 55

EXAMPLE 8 Applying several techniques

Factor $3x^5 - 3x^3 - 24x^2 + 24$.

Solution

STEP 1: The greatest common factor is 3.

$$3x^5 - 3x^3 - 24x^2 + 24 = 3(x^5 - x^3 - 8x^2 + 8)$$

STEP 2A: The resulting four-term polynomial can be factored by grouping.

$$3(x^5 - x^3 - 8x^2 + 8) = 3\big((x^5 - x^3) + (-8x^2 + 8)\big) \quad \text{Associative property}$$
$$= 3\big(x^3(x^2 - 1) - 8(x^2 - 1)\big) \quad \text{Factor out common factors.}$$
$$= 3(x^3 - 8)(x^2 - 1) \quad \text{Factor out } x^2 - 1.$$

STEP 2B: Both binomials in this expression can be factored further. The binomial $x^3 - 8$ can be factored as a difference of two cubes, and the binomial $x^2 - 1$ can be factored as a difference of two squares.

$$3(x^3 - 8)(x^2 - 1) = 3(x - 2)(x^2 + 2x + 4)(x - 1)(x + 1)$$

NOTE: The trinomial $x^2 + 2x + 4$ is prime and cannot be factored further.

STEP 3: The completely factored polynomial is $3(x - 2)(x^2 + 2x + 4)(x - 1)(x + 1)$.

Now Try Exercise 45

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6.5 Putting It All Together

CONCEPT	EXPLANATION	EXAMPLES
Greatest Common Factor	Factor out the greatest common factor, or monomial, that occurs in each term.	$2x^2 - 4x + 10 = 2(x^2 - 2x + 5)$ $3x^3 + 6x = 3x(x^2 + 2)$ $7xy - x^2y = xy(7 - x)$
Factoring by Grouping	Use the associative and distributive properties to factor a polynomial with four terms.	$x^3 - 3x^2 + 2x - 6 = (x^3 - 3x^2) + (2x - 6)$ $= x^2(x - 3) + 2(x - 3)$ $= (x^2 + 2)(x - 3)$
Factoring Binomials	Use the difference of two squares, the difference of two cubes, or the sum of two cubes.	$9x^2 - 4 = (3x - 2)(3x + 2)$ $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$ $x^3 + 27 = (x + 3)(x^2 - 3x + 9)$
Factoring Trinomials	Use FOIL in reverse or grouping.	$x^2 + 5x - 6 = (x + 6)(x - 1)$ Check middle term: $-x + 6x = 5x$. ✓ $4x^2 + \mathbf{4x} - 3 = (4x^2 \mathbf{- 2x}) + (\mathbf{6x} - 3)$ $= 2x(2x - 1) + 3(2x - 1)$ $= (2x + 3)(2x - 1)$

6.5 Exercises

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CONCEPTS AND VOCABULARY

1. What do the letters GCF mean?
2. A good first step for factoring polynomials is to factor out the _____.
3. If a polynomial has four terms, what factoring method might be appropriate?
4. If a polynomial is a binomial, we can try to factor it as a difference of two ____, a difference of two ____, or a sum of two ____.
5. Can $x^2 + 1$ be factored? Explain.
6. Can $x^3 + 1$ be factored? Explain.
7. If a polynomial is a trinomial, we can try to factor it as a perfect ____ trinomial. Otherwise, try factoring it by ____ or apply ____ in reverse.
8. The last step for factoring is to be sure that the polynomial is _____ factored.

WARM UP

Exercises 9–24: Factor completely, if possible.

9. $4x - 2$
10. $x^2 + 3x$
11. $2y^2 - 4y + 4$
12. $5y^2 - 25y + 10$
13. $z^2 - 4$
14. $9z^2 - 25$
15. $a^3 + 8$
16. $8a^3 - 1$
17. $4b^2 - 12b + 9$
18. $b^2 + 4b + 4$
19. $m^2 + 9$
20. $4m^2 + 49$
21. $x^3 - x^2 + 5x - 5$
22. $3x^3 + 6x^2 + x + 2$
23. $y^2 - 5y + 4$
24. $y^2 - 3y - 10$

GENERAL FACTORING

Exercises 25–62: Factor completely.

25. $x^3 + 4x^2 - 9x - 36$
26. $6x^2 - 19x + 15$

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27. $8a^3 - 64$

28. $ab^2 - 4a$

29. $12x^4 - 18x^3 + 4x^2 - 6x$

30. $3x^2y + 24xy + 48y$

31. $54t^4 + 16t$

32. $3t^3 + 18t^2 - 48t$

33. $2r^3 + 6r^2 - 2r - 6$

34. $3r^4 + 3r^3 - 24r - 24$

35. $6z^4 - 21z^3 - 45z^2$

36. $3x^4y + 24xy^4$

37. $12b^4 - 10b^3 + 2b^2$

38. $6a^4b + 4a^3b + 18a^2b + 12ab$

39. $6y^2z - 24z^3$

40. $6y^3z - 48z^4$

41. $3x^2y - 30xy + 75y$

42. $8x^3 + y^3$

43. $27m^3 - 8n^3$

44. $45m^3 - 69m^2 + 12m$

45. $3x^5 - 12x^3 - 3x^2 + 12$

46. $8x^3 - 8$

47. $5a^2 - 27a - 18$

48. $2a^2 - 6ab + 3a - 9b$

49. $3rt^2 + 33rt + 90r$

50. $9t^2 + 24t + 16$

51. $9b^3 + 6b^2 + 12b + 8$

52. $5b^3 - 55b^2 - 60b$

53. $6n^3 + 2n^2 - 10n$

54. $7n^4 + 28n^3 - 63n^2$

55. $4x^2 - 36y^2$

56. $64x^2 - 25y^2$

57. $2a^3 - 16a^2 + 32a$

58. $24a^3 + 72a^2 + 54a$

59. $32xy^3 + 4x$

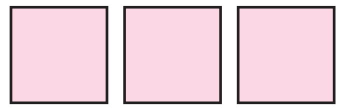
60. $24x^3 - 4x^2 - 160x$

61. $8b^4 + 24b^3 - 2b^2 - 6b$

62. $3z^3 - 6z^2 - 27z + 54$

GEOMETRY

63. ***Dimensions of a Square*** If three identical squares have a total area of $27x^2 + 18x + 3$, find the length of one side of one of the squares.



64. ***Dimensions of a Cube*** If three identical cubes have a total volume of $3x^3 + 18x^2 + 36x + 24$, find the length of one side of one of the cubes.

WRITING ABOUT MATHEMATICS

65. Explain how the number of terms in a polynomial can help determine what method should be used to factor it.

66. Describe a method for determining whether a polynomial has been factored correctly.

6.6 Solving Equations by Factoring I (Quadratics)

The Zero-Product Property • Solving Quadratic Equations • Applications

A LOOK INTO MATH ▶

If a golf ball is hit upward at 132 feet per second, or 90 miles per hour, then its height h in feet above the ground after t seconds is given by $h = 132t - 16t^2$. The expression $132t - 16t^2$ is an example of a *quadratic polynomial*. To determine the elapsed time between when the ball is hit and when it strikes the ground (or when $h = 0$), we solve the *quadratic equation*

$$132t - 16t^2 = 0.$$

One method for solving this equation is by factoring. (See Example 4.) In this section we discuss how to use factoring to solve a variety of equations.

NEW VOCABULARY

- ☐ Zero-product property
- ☐ Zeros (of a polynomial)
- ☐ Quadratic polynomial
- ☐ Quadratic equation
- ☐ Standard form

The Zero-Product Property

To solve equations we often use the **zero-product property**, which states that if the product of two numbers (or expressions) is 0, then at least one of the numbers (or expressions) must equal 0.

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ZERO-PRODUCT PROPERTY

For all real numbers a and b, if $ab = 0$, then $a = 0$ or $b = 0$ (or both).

NOTE: The zero-product property works only for 0. If $ab = 1$, then it does *not* follow that $a = 1$ or $b = 1$. For example, $a = \frac{1}{3}$ and $b = 3$ satisfy the equation $ab = 1$.

STUDY TIP

The zero-product property is used extensively in mathematics for solving many types of equations. Be sure that you learn how to apply this important property correctly.

After factoring an expression, we can use the zero-product property to solve an equation. The left side of the equation

$$3t^2 - 9t = 0$$

may be factored to obtain

$$3t(t - 3) = 0.$$

Note that the product of $3t$ and $t - 3$ is 0. By the zero-product property, either

$$3t = 0 \quad \text{or} \quad t - 3 = 0.$$

Solving each equation for t results in

$$t = 0 \quad \text{or} \quad t = 3.$$

These values can be checked by substituting them into the given equation $3t^2 - 9t = 0$.

$$3(0)^2 - 9(0) = 0 \checkmark \quad \text{Let } t = 0. \text{ It checks.}$$
$$3(3)^2 - 9(3) = 0 \checkmark \quad \text{Let } t = 3. \text{ It checks.}$$

READING CHECK

- How can you tell if a number is a zero of a polynomial?

The t-values of 0 and 3 are called **zeros** of the polynomial $3t^2 - 9t$, because when either is substituted in this polynomial, the result is 0.

EXAMPLE 1 **Applying the zero-product property**

Solve each equation.

(a) $x(x - 1) = 0$ **(b)** $2z^2 = 0$

(c) $(t + 3)(t + 2) = 0$ **(d)** $x(x - 2)(2x + 1) = 0$

Solution

(a) By the zero-product property, $x(x - 1) = 0$ when $x = 0$ or $x - 1 = 0$. The solutions are 0 and 1.

(b) $2z^2 = 2 \cdot z \cdot z$ and $2 \neq 0$, so $2z^2 = 0$ when $z = 0$.

(c) $(t + 3)(t + 2) = 0$ implies that $t + 3 = 0$ or $t + 2 = 0$. The solutions to the given equation are -3 and -2.

(d) We can apply the zero-product property to $x(x - 2)(2x + 1) = 0$. Thus $x = 0$, or $x - 2 = 0$, or $2x + 1 = 0$. The solutions are $-\frac{1}{2}$, 0, and 2.

Now Try Exercises 13, 15, 19, 23

Solving Quadratic Equations

Any **quadratic polynomial** in the variable x can be written as $ax^2 + bx + c$ with $a \neq 0$. Any **quadratic equation** in the variable x can be written as $ax^2 + bx + c = 0$ with $a \neq 0$. This form of quadratic equation is called the **standard form** of a quadratic equation. Table 6.11 on the next page shows examples of quadratic polynomials along with related quadratic equations that can be expressed in standard form.

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TABLE 6.11 Quadratic Polynomials and Equations

Quadratic Polynomial	Quadratic Equation	Standard Form
$x^2 - 2x + 1$	$x^2 - 2x + 1 = 9$	$x^2 - 2x - 8 = 0$
$3x^2 + 7x$	$3x^2 + 7x = 4$	$3x^2 + 7x - 4 = 0$
$x^2 - 9$	$x^2 - 9 = 2$	$x^2 - 11 = 0$
$6x - x^2 + 2$	$6x - x^2 + 2 = 1$	$-x^2 + 6x + 1 = 0$

READING CHECK

- What does it mean for a polynomial to be written so that it contains descending powers of x?

NOTE: When a quadratic polynomial in the variable x is in standard form and we read it from left to right, the terms contain *descending powers* of x. In other words, the first term contains x^2, the second term contains x, and the third term is a constant (the exponent on x is 0).

To solve a quadratic equation we often use factoring and the zero-product property. This method is summarized by the following steps. Although it is not necessary to label each step in the solution to a quadratic equation, it is important to keep these steps in mind.

SOLVING QUADRATIC EQUATIONS

To solve a quadratic equation by factoring, follow these steps.

STEP 1: If necessary, write the equation in standard form as $ax^2 + bx + c = 0$.

STEP 2: Factor the left side of the equation using any method.

STEP 3: Apply the zero-product property.

STEP 4: Solve each of the resulting equations. Check any solutions.

EXAMPLE 2 **Solving equations by factoring**

Solve each quadratic equation. Check your answers.

(a) $x^2 + 2x = 0$ **(b)** $y^2 = 16$ **(c)** $z^2 - 3z + 2 = 0$ **(d)** $2x^2 = 5 - 9x$

Solution

(a) Because $x^2 + 2x = 0$ is in standard form, we begin by factoring out the GCF, x.

$$x^2 + 2x = 0 \quad \text{Given equation}$$
$$x(x + 2) = 0 \quad \text{Factor out } x. \text{ (Step 2)}$$
$$x = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{Zero-product property (Step 3)}$$
$$x = 0 \quad \text{or} \quad x = -2 \quad \text{Solve for } x. \text{ (Step 4)}$$

To check these values, substitute -2 and 0 for x in the given equation.

$$(-2)^2 + 2(-2) \stackrel{?}{=} 0 \qquad (0)^2 + 2(0) \stackrel{?}{=} 0 \quad \text{Substitute } -2 \text{ and } 0.$$
$$0 = 0 \checkmark \qquad 0 = 0 \checkmark \quad \text{Both answers check.}$$

Therefore the solutions are -2 and 0.

(b) To write $y^2 = 16$ in standard form we begin by subtracting 16 from each side to obtain 0 on the right side.

$$y^2 = 16 \quad \text{Given equation}$$
$$y^2 - 16 = 0 \quad \text{Subtract 16. (Step 1)}$$
$$(y - 4)(y + 4) = 0 \quad \text{Difference of squares (Step 2)}$$
$$y - 4 = 0 \quad \text{or} \quad y + 4 = 0 \quad \text{Zero-product property (Step 3)}$$
$$y = 4 \quad \text{or} \quad y = -4 \quad \text{Solve for } y. \text{ (Step 4)}$$

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To check these values, substitute -4 and 4 for y in the given equation.

$$(-4)^2 \stackrel{?}{=} 16 \qquad (4)^2 \stackrel{?}{=} 16 \qquad \text{Substitute } -4 \text{ and } 4.$$
$$16 = 16 \checkmark \qquad 16 = 16 \checkmark \qquad \text{Both answers check.}$$

The solutions are -4 and 4.

(c) We begin by factoring the left side of the equation, $z^2 - 3z + 2$.

$$z^2 - 3z + 2 = 0 \qquad \text{Given equation}$$
$$(z - 1)(z - 2) = 0 \qquad \text{Factor. (Step 2)}$$
$$z - 1 = 0 \quad \text{or} \quad z - 2 = 0 \qquad \text{Zero-product property (Step 3)}$$
$$z = 1 \quad \text{or} \quad z = 2 \qquad \text{Solve for } z\text{. (Step 4)}$$

To check these values, substitute 1 and 2 for z in the given equation.

$$1^2 - 3(1) + 2 \stackrel{?}{=} 0 \qquad 2^2 - 3(2) + 2 \stackrel{?}{=} 0 \qquad \text{Substitute 1 and 2.}$$
$$0 = 0 \checkmark \qquad 0 = 0 \checkmark \qquad \text{Both answers check.}$$

The solutions are 1 and 2.

(d) We write $2x^2 = 5 - 9x$ in standard form by adding -5 and $9x$ to each side.

$$2x^2 = 5 - 9x \qquad \text{Given equation}$$
$$2x^2 + 9x - 5 = 0 \qquad \text{Add } -5 \text{ and } 9x\text{. (Step 1)}$$
$$(2x - 1)(x + 5) = 0 \qquad \text{Factor. (Step 2)}$$
$$2x - 1 = 0 \quad \text{or} \quad x + 5 = 0 \qquad \text{Zero-product property (Step 3)}$$
$$x = \frac{1}{2} \quad \text{or} \quad x = -5 \qquad \text{Solve for } x\text{. (Step 4)}$$

To check these values, substitute -5 and $\frac{1}{2}$ for x in the given equation.

$$2(-5)^2 \stackrel{?}{=} 5 - 9(-5) \qquad 2\left(\frac{1}{2}\right)^2 \stackrel{?}{=} 5 - 9\left(\frac{1}{2}\right) \qquad \text{Substitute } -5 \text{ and } \tfrac{1}{2}.$$
$$50 = 50 \checkmark \qquad \frac{1}{2} = \frac{1}{2} \checkmark \qquad \text{Both answers check.}$$

The solutions are -5 and $\frac{1}{2}$.

Now Try Exercises 27, 35, 43, 49

EXAMPLE 3 Solving an equation by factoring

Solve $6x^2 - x = 12$.

Solution

We *cannot* solve the equation $6x^2 - x = 12$ by factoring out the common factor of x in $6x^2 - x$ and setting each factor equal to 12. Instead we apply the zero-product property by first writing the given equation in the standard form as $ax^2 + bx + c = 0$.

$$6x^2 - x = 12 \qquad \text{Given equation}$$
$$6x^2 - x - 12 = 0 \qquad \text{Subtract 12. (Step 1)}$$
$$(2x - 3)(3x + 4) = 0 \qquad \text{Factor. (Step 2)}$$
$$2x - 3 = 0 \quad \text{or} \quad 3x + 4 = 0 \qquad \text{Zero-product property (Step 3)}$$
$$x = \frac{3}{2} \quad \text{or} \quad x = -\frac{4}{3} \qquad \text{Solve for } x\text{. (Step 4)}$$

The solutions are $-\frac{4}{3}$ and $\frac{3}{2}$.

Now Try Exercise 51

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MAKING CONNECTIONS

Equations and Expressions

The words "equation" and "expression" occur frequently in mathematics. However, they are *not* interchangeable. We often want to *solve* equations to find the values of the variable that make the equation true. We *factor* and *simplify* expressions. An equation is a statement that two expressions are equal. For example, $2x^2 - 3 = 5x + 1$ is an equation where $2x^2 - 3$ and $5x + 1$ are each expressions.

Applications

▶ **REAL-WORLD CONNECTION** To solve application problems we often need to solve equations. The next example illustrates how to solve the application presented in A Look Into Math at the beginning of this section.

EXAMPLE 4 **Modeling the flight of a golf ball**

If a golf ball is hit upward at 132 feet per second, or 90 miles per hour, then its height h in feet after t seconds is $h = 132t - 16t^2$. Find the time when the golf ball strikes the ground.

Solution

The golf ball strikes the ground when its height is 0.

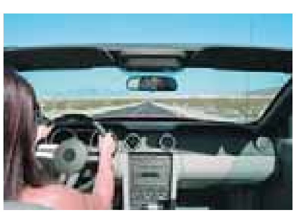
$$\begin{aligned} 132t - 16t^2 &= 0 && \text{Let } h = 0. \\ 4t(33 - 4t) &= 0 && \text{Factor out } 4t. \\ 4t = 0 \quad \text{or} \quad 33 - 4t &= 0 && \text{Zero-product property} \\ t = 0 \quad \text{or} \quad -4t &= -33 && \text{Divide by 4; subtract 33.} \\ t = 0 \quad \text{or} \quad t &= \frac{33}{4} && \text{Solve for } t. \end{aligned}$$

The ball strikes the ground after $\frac{33}{4} = 8.25$ seconds. The solution of 0 is not used in this problem because it corresponds to the time when the ball is hit from ground level.

Now Try Exercise 67(a)

▶ **REAL-WORLD CONNECTION** When you try to stop a car, the faster you are driving, the further the stopping distance. In fact, if you drive twice as fast, the braking distance will be about four times as much, and if you drive three times faster, the braking distance will be about nine times as much.

EXAMPLE 5 **Modeling braking distance**



The braking distance D in feet required to stop a car traveling at x miles per hour on dry, level pavement can be approximated by $D = \frac{1}{11}x^2$. (*Source:* L. Haefner.)

(a) Calculate the braking distance for a car traveling 70 miles per hour.

(b) If the braking distance is 44 feet, calculate the speed of the car.

(c) If you have a calculator available, use it to solve part (b) numerically with a table of values.

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Solution

(a) If $x = 70$, then $D = \frac{1}{11}(70)^2 = \frac{4900}{11} \approx 445$ feet.

(b) ***Symbolic Solution*** Let $D = 44$ in the given equation and solve.

$$\frac{1}{11}x^2 = 44 \quad \text{Let } D = 44.$$

$$\frac{1}{11}x^2 - 44 = 0 \quad \text{Subtract 44.}$$

$$x^2 - 484 = 0 \quad \text{Multiply by 11.}$$

$$(x - 22)(x + 22) = 0 \quad \text{Difference of two squares } (22^2 = 484)$$

$$x - 22 = 0 \quad \text{or} \quad x + 22 = 0 \quad \text{Zero-product property}$$

$$x = 22 \quad \text{or} \quad x = -22 \quad \text{Solve for } x.$$

The car is traveling at (approximately) 22 miles per hour. (Note that $x = -22$ has no physical meaning in this problem.)

(c) ***Numerical Solution*** Let $Y_1 = X\text{^}2/11$ and make a table of values. Scroll through the table, as shown in Figure 6.6, to where $x = 22$ when $y_1 = 44$. Thus the solution is 22 miles per hour.

X	Y_1
20	36.364
21	40.091
22	44
23	48.091
24	52.364
25	56.818
26	61.455

X=22

Figure 6.6

Now Try Exercise 69

▶ **REAL-WORLD CONNECTION** Quadratic equations sometimes are used in applications involving rectangular shapes. The next application involves finding the dimensions of a digital photograph made up of tiny "dots" of color called *pixels*.

EXAMPLE 6 Finding the dimensions of a digital photograph

A small digital photograph is 20 pixels longer than it is wide, as illustrated in Figure 6.7. It has a total of 2400 pixels. Find the dimensions of this photograph.

Dimensions of a Photograph

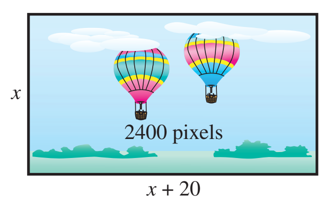

Figure 6.7

Solution

From Figure 6.7 the rectangular photograph has an area of 2400 pixels.

$$x(x + 20) = 2400 \quad \text{Area} = \text{width} \times \text{length}$$

$$x^2 + 20x = 2400 \quad \text{Distributive property}$$

$$x^2 + 20x - 2400 = 0 \quad \text{Subtract 2400.}$$

$$(x - 40)(x + 60) = 0 \quad \text{Factor.}$$

$$x - 40 = 0 \quad \text{or} \quad x + 60 = 0 \quad \text{Zero-product property}$$

$$x = 40 \quad \text{or} \quad x = -60 \quad \text{Solve for } x.$$

The only valid solution is 40. Thus the dimensions of the photograph are 40 pixels by $40 + 20 = 60$ pixels.

Now Try Exercise 73

CRITICAL THINKING

Are the two solutions to $x(2x + 1) = 1$ found by letting $x = 1$ or $2x + 1 = 1$? Explain your reasoning. What are the solutions to this equation?

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6.6 Putting It All Together

CONCEPT	EXPLANATION	EXAMPLES
Zero-Product Property	If the product of two or more expressions is 0, then at least one of the expressions must equal 0.	$ab = 0$ implies that $a = 0$ or $b = 0$. $x(x + 1) = 0$ implies that $x = 0$ or $x + 1 = 0$. $z(z - 1)(z + 2) = 0$ implies that $z = 0$ or $z - 1 = 0$ or $z + 2 = 0$.
Solving Quadratic Equations by Factoring	**1.** Write the equation as $ax^2 + bx + c = 0$. **2.** Factor the left side of this equation. **3.** Apply the zero-product property. **4.** Solve each resulting equation. Check any solutions.	$2x^2 + 11x = 6$ $2x^2 + 11x - 6 = 0$ Step 1 $(2x - 1)(x + 6) = 0$ Step 2 $2x - 1 = 0$ or $x + 6 = 0$ Step 3 $x = \frac{1}{2}$ or $x = -6$ Step 4

6.6 Exercises

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CONCEPTS AND VOCABULARY

1. If $ab = 0$, then either $a =$ _____ or $b =$ _____.

2. Can the zero-product property be used to state that if $(x - 1)(x - 2) = 3$, then either $x - 1 = 3$ or $x - 2 = 3$? Explain your answer.

3. If $2x(x + 6) = 0$, then either _____ or _____.

4. What is a good first step when you are solving the equation $4x^2 + 1 = 4x$ by factoring?

5. What is the next step when you are solving the equation $(x + 5)(x - 4) = 0$?

6. Factoring is an important method for _____ equations.

7. Because $2(4) - 8 = 0$, the value 4 is called a(n) _____ of the polynomial $2x - 8$.

8. What is the zero of the polynomial $3x - 6$?

9. Any quadratic equation in the variable x can be written in standard form as _____.

10. Standard form for $x^2 + 1 = 6x$ is _____.

11. When written in standard form and read from left to right, a quadratic polynomial in the variable x contains _____ powers of x.

12. For the constant term in a quadratic polynomial in the variable x, the exponent on x is _____.

ZERO-PRODUCT PROPERTY

Exercises 13–24: Solve the equation.

13. $x^2 = 0$
14. $5m^2 = 0$
15. $2x(x + 8) = 0$
16. $x(x + 10) = 0$
17. $(y - 1)(y - 2) = 0$
18. $(y + 4)(y - 3) = 0$
19. $(2z - 1)(4z - 3) = 0$
20. $(6z + 5)(z - 7) = 0$
21. $(1 - 3n)(3 - 7n) = 0$
22. $(5 - n)(5 + n) = 0$
23. $x(x - 5)(x - 8) = 0$
24. $x(x + 1)(x - 6) = 0$

SOLVING QUADRATIC EQUATIONS

Exercises 25–60: Solve and check.

25. $x^2 - x = 0$
26. $2x^2 + 4x = 0$

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27. $z^2 - 5z = 0$

28. $6z^2 - 3z = 0$

29. $10y^2 + 15y = 0$

30. $2y^2 + 3y = 0$

31. $x^2 - 1 = 0$

32. $x^2 - 9 = 0$

33. $4n^2 - 1 = 0$

34. $9n^2 - 4 = 0$

35. $z^2 + 3z + 2 = 0$

36. $z^2 - 2z - 3 = 0$

37. $x^2 - 12x + 35 = 0$

38. $x^2 - x - 20 = 0$

39. $2b^2 + 3b - 2 = 0$

40. $3b^2 + b - 2 = 0$

41. $6y^2 + 19y + 10 = 0$

42. $4y^2 - 25y - 21 = 0$

43. $x^2 = 25$

44. $x^2 = 81$

45. $t^2 = 5t$

46. $10t^2 = -5t$

47. $3m^2 = -9m$

48. $4m^2 = 9$

49. $x^2 = 5x + 6$

50. $2x^2 + 3x = 14$

51. $12z^2 = 5 - 4z$

52. $12z^2 + 11z = 15$

53. $t(t + 1) = 2$

54. $t(t - 7) = -12$

55. $x(2x + 5) = 3$

56. $x(3x + 2) = 5$

57. $12x^2 + 12x = -3$

58. $18x^2 + 2 = 12x$

59. $30y^2 + 50y + 20 = 0$

60. $30y^2 - 25y + 5 = 0$

GEOMETRY

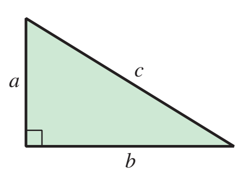
61. ***Dimensions of a Square*** A square has an area of 144 square feet. Find the length of a side.

62. ***Dimensions of a Cube*** A cube has a surface area of 96 square feet. Find the length of a side.

63. ***Radius of a Circle*** The numerical difference between the area and the circumference of a circle is 8π. Find the radius of the circle. (*Hint:* First factor out π in your equation.)

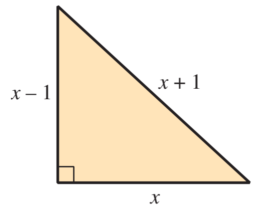
64. ***Dimensions of a Rectangle*** A rectangle is 5 feet longer than it is wide and has an area of 126 square feet. What are its dimensions?

Exercises 65 and 66: ***The Pythagorean Theorem*** *Suppose that a right triangle has legs a and b and hypotenuse c, as illustrated in the figure. Then these values satisfy* $a^2 + b^2 = c^2$.

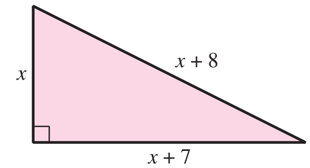


Use the Pythagorean theorem to find the value of x in the figure.

65.



66.



APPLICATIONS

67. ***Flight of a Golf Ball*** (Refer to Example 4.) The height h in feet of a golf ball after t seconds is given by $h = 96t - 16t^2$.

(a) How long does it take for the golf ball to hit the ground?

(b) Make a table of h for $t = 0, 1, 2, \ldots, 6$. After how many seconds does the golf ball reach its maximum height?

68. ***Flight of a Baseball*** The height h in feet of a baseball after t seconds is given by $h = -16t^2 + 88t + 3$. At what values of t is the height of the baseball 75 feet?

69. ***Braking Distance*** (Refer to Example 5.) The braking distance D in feet required to stop a car traveling x miles per hour on dry, level pavement can be approximated by $D = \frac{1}{11}x^2$.

(a) Calculate the braking distance for 30 miles per hour and 60 miles per hour. How do your answers compare?

(b) If the braking distance is 33 feet, estimate the speed of the car.

(c) If you have a calculator, use it to solve part (b) numerically. Do your answers agree?

70. ***Braking Distance*** The braking distance D in feet required to stop a car traveling x miles per hour on wet, level pavement is approximated by $D = \frac{1}{9}x^2$.

(a) Calculate the braking distance for 36 miles per hour and 72 miles per hour. How do your answers compare?

(b) If the braking distance is 49 feet, estimate the speed of the car.

(c) If you have a calculator, use it to solve part (b) numerically. Do your answers agree?

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71. ***Women in the Workforce*** The number of women W in the workforce in millions can be estimated by the equation $W = \frac{19}{3125}x^2 + \frac{11}{2}$, where $x = 0$ corresponds to 1900, $x = 10$ corresponds to 1910, and so on until $x = 100$ corresponds to 2000. (*Source:* U.S. Census Bureau.)

(a) How many women were in the workforce in 1930 and in 2000?

(b) Use a table of values to estimate the year in which 45 million women were in the workforce.

72. ***HIV/AIDS*** Federal funding F in billions of dollars for HIV/AIDS research from 1981 to 2006 can be modeled by

$$F = 0.0316x^2 + 0.015x - 0.046,$$

where $x = 1$ corresponds to 1981, $x = 2$ corresponds to 1982, and so on until $x = 26$ corresponds to 2006. (*Source:* Kaiser Family Foundation.)

(a) Estimate federal funding for HIV/AIDS research in 2001.

(b) Use a table of values to estimate the year in which federal funding for HIV/AIDS research reached \$18.5 billion.

73. ***Digital Photographs*** (Refer to Example 6.) A digital photograph is 10 pixels longer than it is wide and has a total area of 2000 pixels. Find the dimensions of this rectangular photograph.

74. ***Dimensions of a Building*** The rectangular floor of a shed has a length 4 feet longer than its width, and its area is 140 square feet. Let x be the width of the floor.

(a) Write a quadratic equation whose solution gives the width of the floor.

(b) Solve this equation.

WRITING ABOUT MATHEMATICS

75. List four steps for solving a quadratic equation by factoring.

76. Explain why factoring is important.

SECTIONS 6.5 and 6.6

Checking Basic Concepts

1. Factor out the greatest common factor.
(a) $9a^2 - 18a + 27$ **(b)** $7xy^2 + 28x$

2. Factor completely.
(a) $6z^4 - 28z^3 + 16z^2$ **(b)** $2r^2t^2 - 18r^2$

3. Factor completely.
(a) $36x^3 - 48x^2 + 16x$
(b) $24b^3 - 81$

4. Solve each quadratic equation.
(a) $4y^2 - 6y = 0$ **(b)** $5z^2 + 2z = 3$

5. Solve $x^2 + 2x - 3 = 0$ symbolically and numerically with a table of values.

6. If a golf ball is hit upward at 60 miles per hour, then its height h in feet after t seconds is given by $h = 88t - 16t^2$. Use factoring to determine when the golf ball strikes the ground.

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6.7 Solving Equations by Factoring II (Higher Degree)

Polynomials with Common Factors ● Special Types of Polynomials

A LOOK INTO MATH ▶

In this section we discuss factoring polynomials having higher degree. Polynomials of degree 2 or higher are often used in applications. For example, the polynomial

$$0.0013x^3 - 0.085x^2 + 1.6x + 12$$

models natural gas consumption in the United States in trillions of cubic feet, where $x = 0$ corresponds to 1960, $x = 10$ corresponds to 1970, and so on until $x = 40$ corresponds to 2000, as shown in Figure 6.8. (This trend did not continue after 2000.) In Exercises 79–82 we discuss further the consumption of natural gas. (*Source:* Department of Energy.)

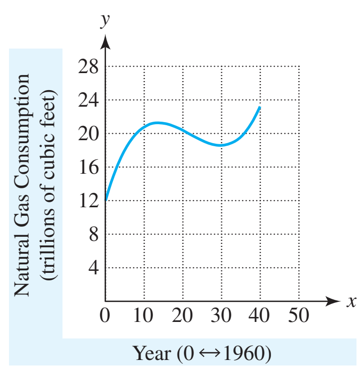

Figure 6.8

READING CHECK

- What is a good first step in factoring a polynomial?

Polynomials with Common Factors

The first step in factoring a polynomial is to factor out the greatest common factor (GCF). Once this is accomplished, the resulting expression should be factored further by using any of the factoring methods discussed earlier in this chapter.

EXAMPLE 1 **Factoring trinomials with common factors**

Factor each trinomial completely.

(a) $-4x^2 + 28x - 40$ **(b)** $10x^3 + 28x^2 - 6x$

Solution

(a) Each term in $-4x^2 + 28x - 40$ has a factor of -4, so we start by factoring out -4. (In this example, we factor out the *opposite* of the GCF to obtain a positive leading coefficient in the resulting expression.)

$$-4x^2 + 28x - 40 = -4(x^2 - 7x + 10) \quad \text{Factor out } -4.$$
$$= -4(x - 5)(x - 2) \quad \text{Factor the trinomial.}$$

(b) We start by factoring out the GCF for $10x^3 + 28x^2 - 6x$, which is $2x$.

$$10x^3 + 28x^2 - 6x = 2x(5x^2 + 14x - 3) \quad \text{Factor out } 2x.$$
$$= 2x(5x - 1)(x + 3) \quad \text{Factor the trinomial.}$$

Now Try Exercises 13, 21

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READING CHECK

- When solving a polynomial equation by factoring, to which factors can we apply the zero-product property?

Many equations involving higher degree polynomials can be solved using the four-step process discussed in Section 6.6. It is important to remember that the zero-product property applies to *all* factors that contain the variable. In the next example we apply the zero-product property to three factors, resulting in three solutions.

EXAMPLE 2 **Solving polynomial equations**

Solve each equation.

(a) $x^3 - x^2 - 6x = 0$ **(b)** $4x^4 + 10x^3 = 6x^2$

Solution

(a) We start by factoring out the GCF, which is x.

$x^3 - x^2 - 6x = 0$	Given equation
$x(x^2 - x - 6) = 0$	Factor out x.
$x(x - 3)(x + 2) = 0$	Factor the trinomial.
$x = 0$ or $x - 3 = 0$ or $x + 2 = 0$	Zero-product property
$x = 0$ or $x = 3$ or $x = -2$	Solve for x.

The solutions are -2, 0, and 3.

(b) We start by subtracting $6x^2$ from each side to obtain 0 on one side of the equation.

$4x^4 + 10x^3 = 6x^2$	Given equation
$4x^4 + 10x^3 - 6x^2 = 0$	Subtract $6x^2$.
$2x^2(2x^2 + 5x - 3) = 0$	Factor out the GCF, $2x^2$.
$2x^2(2x - 1)(x + 3) = 0$	Factor the trinomial.
$2x \cdot x = 0$ or $2x - 1 = 0$ or $x + 3 = 0$	Zero-product property
$x = 0$ or $x = \frac{1}{2}$ or $x = -3$	Solve for x.

The solutions are -3, 0, and $\frac{1}{2}$.

Now Try Exercises 59, 65

STUDY TIP

If you are having difficulty with your studies, you may be able to find help at the student support services office on your campus.

CRITICAL THINKING

Calculate the outside surface area A of the box shown in Figure 6.9 two different ways.

▶ **REAL-WORLD CONNECTION** The corners of a square piece of metal are cut out to form a box, as shown in Figure 6.9. This square piece of metal has sides with length 10 inches, and the cutout corners are squares with length x. The outside surface area A of this box, including the bottom and the sides but *not* the top, is $A = 100 - 4x^2$. (See Critical Thinking in the margin.)

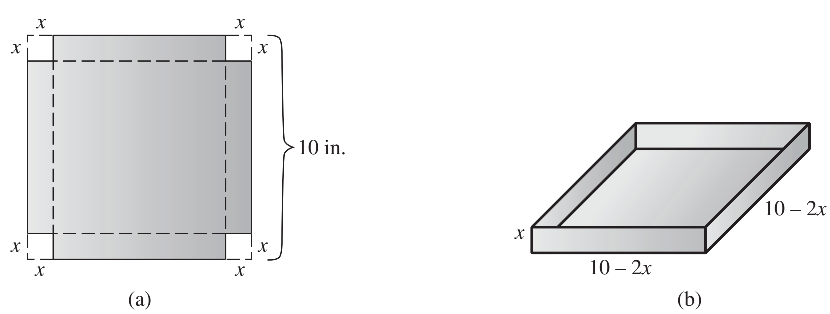

Figure 6.9

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EXAMPLE 3 **Finding a dimension of a box**

Find the value of x in Figure 6.9 if the outside surface area A is 84 square inches.

Solution

Let $A = 84$ in the equation $A = 100 - 4x^2$ and solve for x.

$$100 - 4x^2 = 84 \quad \text{Let } A = 84.$$
$$16 - 4x^2 = 0 \quad \text{Subtract 84.}$$
$$4(4 - x^2) = 0 \quad \text{Factor out 4.}$$
$$4(2 - x)(2 + x) = 0 \quad \text{Difference of two squares}$$
$$2 - x = 0 \quad \text{or} \quad 2 + x = 0 \quad \text{Zero-product property}$$
$$x = 2 \quad \text{or} \quad x = -2 \quad \text{Solve for } x.$$

If squares measuring 2 inches on a side are cut out of each corner, the surface area of the box is 84 square inches. (Note that $x = -2$ has no physical meaning in this problem.)

Now Try Exercise 77

Special Types of Polynomials

Some types of polynomials of higher degree can be factored by using methods that were presented earlier in this chapter.

EXAMPLE 4 **Factoring higher degree polynomials**

Factor each polynomial completely.

(a) $x^4 - 16$ **(b)** $y^4 + 5y^2 + 4$ **(c)** $x^4 + 2x^2y^2 + y^4$ **(d)** $r^4 - t^4$

Solution

(a) We view this polynomial as the difference of two squares, where $x^4 = (x^2)^2$ and $16 = 4^2$. Then we factor twice.

$$x^4 - 16 = (x^2)^2 - 4^2 \quad \text{Rewrite.}$$
$$= (x^2 - 4)(x^2 + 4) \quad \text{Difference of two squares}$$
$$= (x - 2)(x + 2)(x^2 + 4) \quad \text{Difference of two squares}$$

Note that $x^2 + 4$ does not factor.

(b) Because the trinomial $a^2 + 5a + 4$ factors as $(a + 1)(a + 4)$, we let $a = y^2$ and then factor the given trinomial.

$$y^4 + 5y^2 + 4 = (y^2)^2 + 5(y^2) + 4$$
$$= (y^2 + 1)(y^2 + 4)$$

Note that neither $y^2 + 1$ nor $y^2 + 4$ can be factored further.

(c) Because the perfect square trinomial $a^2 + 2ab + b^2$ factors as $(a + b)^2$, we let $a = x^2$ and $b = y^2$ and then factor the given trinomial.

$$x^4 + 2x^2y^2 + y^4 = (x^2)^2 + 2x^2y^2 + (y^2)^2$$
$$= (x^2 + y^2)^2$$

(d) Because the difference of squares $a^2 - b^2$ factors as $(a - b)(a + b)$, we let $a = r^2$ and $b = t^2$ and then factor the given binomial.

$$r^4 - t^4 = (r^2)^2 - (t^2)^2$$
$$= (r^2 - t^2)(r^2 + t^2)$$
$$= (r - t)(r + t)(r^2 + t^2)$$

Note that $r^2 + t^2$ cannot be factored.

Now Try Exercises 29, 37, 43, 45

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EXAMPLE 5 **Solving an equation**

Solve $x^5 - 81x = 0$.

Solution

We start by factoring out the common factor of x.

$x^5 - 81x = 0$	Given equation
$x(x^4 - 81) = 0$	Factor out x.
$x(x^2 - 9)(x^2 + 9) = 0$	Difference of two squares
$x(x - 3)(x + 3)(x^2 + 9) = 0$	Difference of two squares
$x = 0$ or $x - 3 = 0$ or $x + 3 = 0$ or $x^2 + 9 = 0$	Zero-product property
$x = 0$ or $x = 3$ or $x = -3$	Solve for x.

Note that $x^2 + 9 = 0$ has no real-number solutions because the square of a number plus 9 is never 0. The solutions are -3, 0, and 3.

Now Try Exercise 63

6.7 Putting It All Together

CONCEPT	EXPLANATION	EXAMPLES
Common Factors	A first step when factoring polynomials is to factor out the GCF. Once this is done, factor the resulting expression, if possible.	$x^3 - 4x^2 - 5x = x(x^2 - 4x - 5)$ $= x(x - 5)(x + 1)$ $4x^4 - 16x^2 = 4x^2(x^2 - 4)$ $= 4x^2(x - 2)(x + 2)$
Factoring Higher Degree Polynomials	Some higher degree polynomials can be factored by using the same methods that we use to factor quadratic polynomials.	$x^4 + 6x^2 + 5 = (x^2 + 5)(x^2 + 1)$ $4y^4 - 25 = (2y^2 - 5)(2y^2 + 5)$ $2z^4 - 32 = 2(z^4 - 16)$ $= 2(z^2 - 4)(z^2 + 4)$ $= 2(z - 2)(z + 2)(z^2 + 4)$
Solving Equations by Factoring	Use factoring and the zero-product property to solve polynomial equations.	$3x^3 - 12x^2 + 9x = 0$ $3x(x^2 - 4x + 3) = 0$ $3x(x - 3)(x - 1) = 0$ $3x = 0$ or $x - 3 = 0$ or $x - 1 = 0$ $x = 0$ or $x = 3$ or $x = 1$ The solutions are 0, 1, and 3.

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6.7 Exercises

CONCEPTS AND VOCABULARY

1. When you are factoring polynomials, a good first step is to factor out the _____.

2. When you are solving an equation by factoring, the _____ property is used.

3. The zero-product property applies to all _____ containing the variable.

4. Because $a^2 - 2ab + b^2 = (a - b)^2$, it follows that $x^4 - 2x^2y^2 + y^4 =$ _____.

5. Because $x^2 + 3x + 2 = (x + 1)(x + 2)$, it follows that $z^4 + 3z^2 + 2 =$ _____.

6. If $x^4 - 1$ is factored as $(x^2 - 1)(x^2 + 1)$, is it factored *completely*? Explain.

7. If $x^3 - x^2 + x - 1$ is factored as $(x - 1)(x^2 + 1)$, is it factored *completely*? Explain.

8. When you are solving the equation $x^3 = x$, what is a good first step?

9. When you are solving $x^4 - x^2 = 0$, what is a good first step?

10. How many real-number solutions does the equation $(x - 6)(x^2 + 4) = 0$ have?

FACTORING POLYNOMIALS

Exercises 11–48: Factor the polynomial completely.

11. $5x^2 - 5x - 30$

12. $3x^2 - 15x + 12$

13. $-4y^2 - 32y - 48$

14. $-7y^2 + 14y + 21$

15. $-20z^2 - 110z - 50$

16. $-12z^2 - 54z + 30$

17. $60 - 64t - 28t^2$

18. $18 - 45t - 27t^2$

19. $r^3 - r$

20. $r^3 + 2r^2 - 3r$

21. $3x^3 + 3x^2 - 18x$

22. $6x^3 - 26x^2 - 20x$

23. $72z^3 + 12z^2 - 24z$

24. $6z^3 - 4z^2 - 42z$

25. $x^4 - 4x^2$

26. $4x^4 - 36x^2$

27. $t^4 + t^3 - 2t^2$

28. $t^4 + 5t^3 - 24t^2$

29. $x^4 - 5x^2 + 6$

30. $x^4 - 3x^2 - 10$

31. $2x^4 + 7x^2 + 3$

32. $3x^4 - 8x^2 + 5$

33. $y^4 + 6y^2 + 9$

34. $y^4 - 10y^2 + 25$

35. $x^4 - 9$

36. $x^4 - 25$

37. $x^4 - 81$

38. $4x^4 - 64$

39. $z^5 + 2z^4 + z^3$

40. $6z^5 - 47z^4 + 35z^3$

41. $2x^2 + xy - y^2$

42. $2x^2 + 5xy + 2y^2$

43. $a^4 - 2a^2b^2 + b^4$

44. $a^3 + 2a^2b + ab^2$

45. $x^3 - xy^2$

46. $2x^2y - 2y^3$

47. $4x^3 + 4x^2y + xy^2$

48. $x^2y - 6xy^2 + 9y^3$

SOLVING EQUATIONS

Exercises 49–54: Do the following.

49. **(a)** Factor $x^3 - 4x$.
(b) Solve $x^3 - 4x = 0$.

50. **(a)** Factor $4x^3 - 16x$.
(b) Solve $4x^3 - 16x = 0$.

51. **(a)** Factor $2y^3 - 6y^2 - 36y$.
(b) Solve $2y^3 - 6y^2 - 36y = 0$.

52. **(a)** Factor $z^4 - 13z^2 + 36$.
(b) Solve $z^4 - 13z^2 + 36 = 0$.

53. **(a)** Factor $x^3 - x^2 + 4x - 4$.
(b) Solve $x^3 - x^2 + 4x - 4 = 0$.

54. **(a)** Factor $y^4 - 8y^2 + 16$.
(b) Solve $y^4 - 8y^2 + 16 = 0$.

Exercises 55–76: Solve.

55. $3x^2 + 33x + 72 = 0$

56. $4x^2 - 16x - 20 = 0$

57. $25x^2 = 50x + 75$

58. $10x^2 = 20x + 80$

59. $y^3 - 3y^2 - 4y = 0$

60. $y^3 - 3y^2 + 2y = 0$

61. $3z^3 + 6z^2 = 72z$

62. $4z^3 = 4z^2 + 24z$

63. $x^4 - 36x^2 = 0$

64. $4x^4 = 100x^2$

65. $r^4 + 6r^3 = 7r^2$

66. $r^4 + 30r^2 = 11r^3$

67. $x^4 - 13x^2 = -36$

68. $x^4 - 17x^2 + 16 = 0$

69. $x^4 + 1 = 2x^2$

70. $x^4 - 8x^2 + 16 = 0$

71. $a^4 = 81$

72. $b^3 = -8$

73. $x^3 - 2x^2 - x + 2 = 0$

74. $x^3 - x^2 + 4x - 4 = 0$

75. $x^3 - 5x^2 + x - 5 = 0$

76. $3x^3 + 2x^2 - 27x - 18 = 0$

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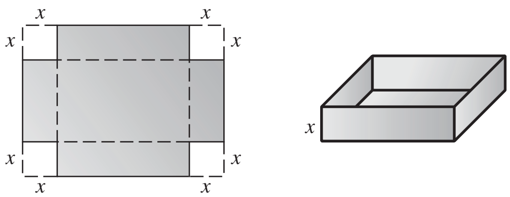
APPLICATIONS

77. ***Dimensions of a Box*** (Refer to Example 3.) A box is made from a rectangular piece of metal with length 20 inches and width 15 inches. The box has no top.

(a) What are the limitations on the size of x? Explain your answer.

(b) Write an expression that gives the outside surface area of the box. (*Hint:* Consider the size of the metal sheet and how much was cut out.)

(c) If the outside surface area of the box is 275 square inches, find x.



78. ***Dimensions of a Box*** Refer to the previous exercise.

(a) Find a polynomial that gives the volume of the box for a given x.

(b) Factor your polynomial completely.

(c) What are the zeros of your polynomial? What do they represent in this problem?

Exercises 79–82: ***U.S. Natural Gas Consumption*** *(Refer to A Look Into Math for this section.) The polynomial*

$$0.0013x^3 - 0.085x^2 + 1.6x + 12$$

models natural gas consumption in trillions of cubic feet, where $x = 0$ corresponds to 1960, $x = 1$ corresponds to 1961, and so on until $x = 40$ corresponds to 2000.

79. How much natural gas was consumed in 1990?

80. In which year between 1970 and 1990 was natural gas consumption about 20.4 trillion cubic feet?

81. Explain any difficulties encountered when you try to solve the equation

$$0.0013x^3 - 0.085x^2 + 1.6x + 12 = 23.2.$$

82. How might you solve the equation in Exercise 81 without factoring? If you were to find the solution to this equation, what would it represent?

WRITING ABOUT MATHEMATICS

83. Compare factoring the polynomial $x^2 + 6x + 5$ with factoring the polynomial $z^4 + 6z^2 + 5$.

84. Suppose that a polynomial can be factored. Explain how its factors can be used to find the zeros of the polynomial. Give an example.

SECTION 6.7 Checking Basic Concepts

1. Factor the trinomial completely.
(a) $3x^2 - 6x - 24$ **(b)** $-10y^2 + 5y + 5$

2. Factor the binomial completely.
(a) $z^4 - 25$ **(b)** $7t^4 - 7$

3. Factor.
(a) $x^4 - 8x^2 + 16$ **(b)** $2y^3 + 17y^2 - 30y$

4. Solve $t^4 + t^3 = 12t^2$.

5. Solve $x^3 - 3x^2 + 2x - 6 = 0$.

CHAPTER 6 Summary

SECTION 6.1 ■ INTRODUCTION TO FACTORING

Terms Related to Factoring Polynomials

Factoring — Writing a polynomial as a product, usually of lower degree polynomials

Common Factor — An expression that is a factor of each term in a polynomial

Example: Some common factors of $4x^4 + 8x^2$ are $2x$, x^2, and $4x^2$.

Greatest Common Factor (GCF) — The common factor with the greatest (integer) coefficient and the highest degree

Example: The GCF of $4x^4 + 8x^2$ is $4x^2$.

$$4x^4 + 8x^2 = 4x^2(x^2 + 2)$$

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Factoring by Grouping Used to factor a four-term polynomial into two binomials

Example: $x^3 + 5x^2 + 3x + 15 = x^2(x + 5) + 3(x + 5)$
$= (x^2 + 3)(x + 5)$

SECTION 6.2 ■ FACTORING TRINOMIALS I $(x^2 + bx + c)$

Review of the FOIL Method A method used for multiplying two binomials

First Terms $(2x + 3)(5x + 4)$: $2x \cdot 5x = 10x^2$

Outside Terms $(2x + 3)(5x + 4)$: $2x \cdot 4 = 8x$

Inside Terms $(2x + 3)(5x + 4)$: $3 \cdot 5x = 15x$

Last Terms $(2x + 3)(5x + 4)$: $3 \cdot 4 = 12$

The product is the sum of these four terms:

$$(2x + 3)(5x + 4) = 10x^2 + 8x + 15x + 12 = 10x^2 + 23x + 12.$$

Factoring Trinomials Having a Leading Coefficient of 1 To factor the trinomial $x^2 + bx + c$, find two numbers, m and n, that satisfy

$$m \cdot n = c \quad \text{and} \quad m + n = b.$$

Then $x^2 + bx + c = (x + m)(x + n)$.

Example: Because $-3 \cdot 5 = -15$ and $-3 + 5 = 2$,

$$x^2 + 2x - 15 = (x - 3)(x + 5).$$

SECTION 6.3 ■ FACTORING TRINOMIALS II $(ax^2 + bx + c)$

Factoring Trinomials by Grouping To factor $ax^2 + bx + c$, perform the following steps. (Assume that a, b, and c are integers and have no factor in common.)

1. Find two numbers, m and n, such that $mn = ac$ and $m + n = b$.
2. Write the trinomial as $ax^2 + mx + nx + c$.
3. Use grouping to factor this expression into two binomials.

Example: To factor $3x^2 + 10x - 8$, find two numbers whose product is -24 and whose sum is 10. These two numbers are $m = 12$ and $n = -2$, so write $10x$ as $12x - 2x$.

$3x^2 + 10x - 8 = 3x^2 + 12x - 2x - 8$ Write $10x$ as $12x - 2x$.

$= (3x^2 + 12x) + (-2x - 8)$ Associative property

$= 3x(x + 4) - 2(x + 4)$ Factor out common factors.

$= (3x - 2)(x + 4)$ Factor out $(x + 4)$.

Factoring with FOIL in Reverse Use trial and error and FOIL in reverse to find the factors of a trinomial.

Example: To factor $3x^2 + 10x - 8$, first find factors of $3x^2$.

$$(3x + \_\_\_)(x + \_\_\_)$$

Then place factors of -8 so that the resulting middle term is $10x$.

$$(3x + -2) \cdot (x + 4)$$

$-2x$

$+ 12x$

$10x$ ✓ ← Middle term checks.

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SECTION 6.4 ■ SPECIAL TYPES OF FACTORING

Difference of Two Squares

$$a^2 - b^2 = (a - b)(a + b)$$

Examples: $x^2 - 16 = (x - 4)(x + 4)$ $(a = x, b = 4)$

$4r^2 - 9t^2 = (2r - 3t)(2r + 3t)$ $(a = 2r, b = 3t)$

Perfect Square Trinomials

$$a^2 + 2ab + b^2 = (a + b)^2 \quad \text{and} \quad a^2 - 2ab + b^2 = (a - b)^2$$

Examples: $4x^2 + 4x + 1 = (2x)^2 + 2(2x)1 + 1^2 = (2x + 1)^2$ $(a = 2x, b = 1)$

$x^2 - 10x + 25 = x^2 - 2 \cdot x \cdot 5 + 5^2 = (x - 5)^2$ $(a = x, b = 5)$

Sums and Differences of Two Cubes

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Examples: $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$ $(a = x, b = 2)$

$27x^3 - 1 = (3x - 1)(9x^2 + 3x + 1)$ $(a = 3x, b = 1)$

SECTION 6.5 ■ SUMMARY OF FACTORING

Guidelines for Factoring Polynomials The following guidelines can be used to factor polynomials in general.

STEP 1: Factor out the greatest common factor, if possible.

STEP 2: **A.** If the polynomial has *four terms*, try factoring by grouping.

B. If the polynomial is a *binomial,* try one of the following.

1. $a^2 - b^2 = (a - b)(a + b)$ Difference of two squares
2. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ Difference of two cubes
3. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ Sum of two cubes

C. If the polynomial is a *trinomial*, check for a perfect square.

1. $a^2 + 2ab + b^2 = (a + b)^2$ Perfect square trinomial
2. $a^2 - 2ab + b^2 = (a - b)^2$ Perfect square trinomial

Otherwise, try to factor the trinomial by grouping or apply FOIL in reverse, as described in Sections 6.2 and 6.3.

STEP 3: Check to make sure that the polynomial is *completely* factored.

Examples: $12x^3 - 12x^2 + 3x = 3x(4x^2 - 4x + 1) = 3x(2x - 1)^2$ Steps 1, 2C, and 3

$9x^3 - 6x^2 + 18x - 12 = 3(3x^3 - 2x^2 + 6x - 4) = 3(x^2 + 2)(3x - 2)$ Steps 1, 2A, and 3

$16x^3 - 100x = 4x(4x^2 - 25) = 4x(2x - 5)(2x + 5)$ Steps 1, 2B, and 3

SECTION 6.6 ■ SOLVING EQUATIONS BY FACTORING I (QUADRATICS)

Zero-Product Property

For any real numbers a and b, if $ab = 0$, then $a = 0$ or $b = 0$ (or both). The zero-product property is used to solve equations.

Examples: $xy = 0$ implies that $x = 0$ or $y = 0$.

$(x + 5)(x - 3) = 0$ implies $x + 5 = 0$ or $x - 3 = 0$.

Zero of a Polynomial A number a is a zero of a polynomial if the result is 0 when a is substituted in that polynomial.

Example: The number -2 is a zero of $x^2 - 4$ because $(-2)^2 - 4 = 0$.

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Solving Quadratic Equations by Factoring To solve a quadratic equation by factoring, follow these steps.

STEP 1: If necessary, use algebra to write the equation as $ax^2 + bx + c = 0$.

STEP 2: Factor the left side of the equation using any method.

STEP 3: Apply the zero-product property.

STEP 4: Solve each of the resulting equations. Check any solutions.

Example:

$x^2 + 7x = 8$ Given equation

$x^2 + 7x - 8 = 0$ Step 1

$(x + 8)(x - 1) = 0$ Step 2

$x + 8 = 0$ or $x - 1 = 0$ Step 3

$x = -8$ or $x = 1$ Step 4

SECTION 6.7 ■ SOLVING EQUATIONS BY FACTORING II (HIGHER DEGREE)

Factoring Polynomials of Higher Degree The distributive property and the techniques for factoring quadratic polynomials can also be applied to polynomials of higher degree.

Examples: $10r^3 + 15r = 5r(2r^2 + 3)$ (To check, multiply the right side.)

Because $2x^2 + x - 1 = (x + 1)(2x - 1)$,

it follows that $2z^4 + z^2 - 1 = (z^2 + 1)(2z^2 - 1)$.

Solving Equations by Factoring Use algebra to obtain 0 on one side of the equation. Factor the other side and apply the zero-product property.

Example:

$x^3 = 4x$ Given equation

$x^3 - 4x = 0$ Subtract $4x$.

$x(x^2 - 4) = 0$ Factor out the GCF, x.

$x(x - 2)(x + 2) = 0$ Difference of two squares

$x = 0$ or $x - 2 = 0$ or $x + 2 = 0$ Zero-product property

$x = 0$ or $x = 2$ or $x = -2$ Solve for x.

CHAPTER 6 Review Exercises

SECTION 6.1

Exercises 1–6: Identify the greatest common factor for the expression and then factor the expression.

1. $5x - 15$

2. $y^2 + 2y$

3. $8z^3 - 4z^2$

4. $6x^4 + 3x^3 - 12x^2$

5. $9xy + 15yz^2$

6. $a^2b^3 + a^3b^2$

Exercises 7–14: Use grouping to factor the given polynomial completely.

7. $x(x + 2) - 3(x + 2)$

8. $y^2(x - 5) + 3y(x - 5)$

9. $z^3 - 2z^2 + 5z - 10$

10. $t^3 + t^2 + 8t + 8$

11. $x^3 - 3x^2 + 6x - 18$

12. $ax + bx - ay - by$

13. $x^5 + 3x^4 - 2x^3 - 6x^2$

14. $2y^4 + 6y^3 + 2y^2 + 6y$

SECTION 6.2

Exercises 15–18: Find an integer pair that has the given product and sum.

15. Product: 20 Sum: 9

16. Product: -21 Sum: 4

17. Product: 36 Sum: -13

18. Product: -100 Sum: -21

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Exercises 19–32: Factor the trinomial completely. If the trinomial cannot be factored, write "prime."

19. $x^2 - x - 12$
20. $x^2 + 10x + 24$
21. $x^2 + 6x - 16$
22. $x^2 - x - 42$
23. $x^2 + 4x - 6$
24. $x^2 - 5x + 8$
25. $x^2 + 2x - 3$
26. $x^2 + 22x + 120$
27. $2x^3 + 6x^2 - 20x$
28. $x^4 - 3x^3 - 28x^2$
29. $10 - 7x + x^2$
30. $24 + 2x - x^2$
31. $-2x^2 - 4x + 30$
32. $-x^3 - 9x^2 + 10x$

SECTION 6.3

Exercises 33–44: Factor the trinomial completely. If the trinomial cannot be factored, write "prime."

33. $9x^2 + 3x - 2$
34. $2x^2 + 3x - 5$
35. $3x^2 + 14x + 15$
36. $35x^2 - 2x - 1$
37. $3x^2 + 4x - 5$
38. $4x^2 - 12x - 5$
39. $24x^2 - 7x - 5$
40. $4x^2 + 33x - 27$
41. $12x^3 + 48x^2 + 21x$
42. $8x^4 + 14x^3 - 30x^2$
43. $12 - 5x - 2x^2$
44. $1 + 3x - 10x^2$

SECTION 6.4

Exercises 45–58: Factor completely.

45. $z^2 - 4$
46. $9z^2 - 64$
47. $36 - y^2$
48. $100a^2 - 81b^2$
49. $x^2 + 14x + 49$
50. $x^2 - 10x + 25$
51. $4x^2 - 12x + 9$
52. $9x^2 + 48x + 64$
53. $8t^3 - 1$
54. $27r^3 + 8t^3$
55. $2x^3 - 50x$
56. $24x^3 + 81$
57. $2x^3 + 28x^2 + 98x$
58. $2x^4 - 128x$

SECTION 6.5

Exercises 59–68: Factor completely.

59. $12x - 8$
60. $6x^3 + 9x^2$
61. $9y^2 - 6y + 6$
62. $yz^2 - 9y$
63. $x^4 + 7x^3 - 4x^2 - 28x$
64. $12x^3 + 36x^2 + 27x$
65. $3ab^3 - 24a$
66. $5x^3 + 20x$
67. $24x^3 - 6xy^2$
68. $x^3y + 27y$

SECTION 6.6

Exercises 69–80: Solve the equation.

69. $mn = 0$
70. $y^2 = 0$
71. $(4x - 3)(x + 9) = 0$
72. $(1 - 4x)(6 + 5x) = 0$
73. $z(z - 1)(z - 2) = 0$
74. $z^2 - 7z = 0$
75. $y^2 - 64 = 0$
76. $y^2 + 9y + 14 = 0$
77. $x^2 = x + 6$
78. $10x^2 + 11x = 6$
79. $t(t - 14) = 72$
80. $t(2t - 1) = 10$

SECTION 6.7

Exercises 81–90: Factor completely.

81. $5x^2 - 15x - 50$
82. $-3x^2 - 6x + 45$
83. $y^3 - 4y$
84. $3y^3 + 6y^2 - 9y$
85. $2z^4 + 14z^3 + 20z^2$
86. $8z^4 - 32z^2$
87. $x^4 - 6x^2 + 9$
88. $2x^4 - 15x^2 - 27$
89. $a^2 + 10ab + 25b^2$
90. $x^3 - xy^2$

Exercises 91–98: Solve.

91. $16x^2 - 72x - 40 = 0$
92. $2x^3 - 11x^2 + 15x = 0$
93. $t^3 = 25t$
94. $t^4 - 7t^3 + 12t^2 = 0$
95. $z^4 + 16 = 8z^2$
96. $z^4 - 256 = 0$
97. $y^3 = -64$
98. $y^3 - y^2 - y + 1 = 0$

APPLICATIONS

99. A square has area $9x^2 + 42x + 49$. Find the length of a side. Make a sketch of the square.

100. A rectangle has area $x^2 + 6x + 5$. Find possible dimensions for the rectangle. Make a sketch of the rectangle.

101. A cube has surface area $6x^2 + 12x + 6$. Find the length of a side.

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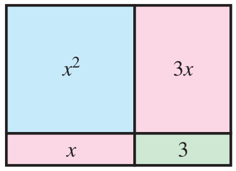
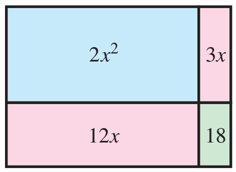
102. Write a polynomial in factored form that represents the total area of the rectangle.



103. Write a polynomial in factored form that represents the total area of the rectangle.



104. ***Radius of a Circle*** The area and the circumference of a circle are numerically equal. Find the radius of the circle.

105. ***Dimensions of a Shed*** The floor of a rectangular shed is 7 feet longer than it is wide and has an area of 120 square feet. What are its dimensions?

106. ***Flight of a Ball*** A ball is hit upward. Its height h in feet after t seconds is $h = -16t^2 + 80t + 4$. At what times is the ball 100 feet in the air?

107. ***Stopping Distance*** The distance D in feet that it takes to stop a car traveling x miles per hour on wet, level pavement can be approximated by $D = \frac{1}{9}x^2 + \frac{11}{3}x$.

(a) Estimate the distance required for the car to stop when it is traveling 45 miles per hour.

(b) If the stopping distance is 80 feet, what is the speed of the car?

(c) If you have a calculator, use it to solve part (b) numerically with a table of values. Do your answers agree?

108. ***Revenue*** A company makes tops for the boxes of pickup trucks. The total revenue R in dollars from selling the tops for p dollars each is given by $R = p(200 - p)$, where $p \leq 200$.

(a) Find R when $p = \$100$.

(b) Find p when $R = \$7500$.

(c) If you have a calculator, use it to solve part (b) numerically with a table of values. Do your answers agree?

109. ***Airline Passengers*** The number N of worldwide airline passengers in millions from 1950 to 2000 is approximated by

$$N = 0.68y^2 + 3.8y + 24,$$

where $y = 0$ corresponds to 1950, $y = 1$ corresponds to 1951, and so on until $y = 50$ corresponds to 2000.

(a) Estimate the number of airline passengers in 1970.

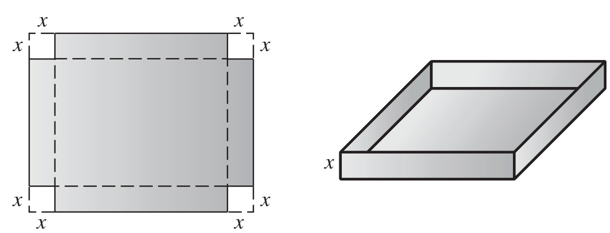
(b) Use a table of values to estimate the year in which the number of airline passengers reached 544 million.

110. ***Digital Photographs*** A digital photograph is 30 pixels longer than it is wide and has a total area of 4000 pixels. Find the dimensions of this photograph.

111. ***Dimensions of a Box*** A box is made from a rectangular piece of metal with length 50 inches and width 40 inches by cutting out square corners of length x and folding up the sides.

(a) Write an expression that gives the surface area of the inside of the box.

(b) If the surface area of the box is 1900 square inches, find x.



112. ***Dimensions of a Cube*** If two identical cubes have a total surface area of $12x^2 + 48x + 48$, find the length of one side of one of the cubes.

CHAPTER 6 Test

CHAPTER Test Prep VIDEOS Step-by-step test solutions are found on the Chapter Test Prep Videos available in MyMathLab and on YouTube (search "RockswoldComboAlg" and click on "Channels").

Exercises 1 and 2: Identify the greatest common factor for the expression. Then factor the expression.

1. $4x^2y - 20xy^2 + 12xy$ **2.** $9a^3b^2 + 3a^2b^2$

Exercises 3 and 4: Factor by grouping.

3. $ay + by + az + bz$ **4.** $3x^3 + x^2 - 15x - 5$

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Exercises 5–10: Factor the trinomial completely. If the trinomial cannot be factored, write "prime."

5. $y^2 + 4y - 12$ **6.** $4x^2 + 20x + 25$

7. $4z^2 - 19z + 12$ **8.** $21 - 17t + 2t^2$

9. $x^2 + 7x - 10$ **10.** $3y^2 + 4y + 2$

Exercises 11–16: Factor completely.

11. $6x^3 + 3x^2 - 3x$ **12.** $2z^4 - 12z^2 - 54$

13. $36y^3 - 100y$ **14.** $7x^4 + 56x$

15. $16a^4 + 24a^3 + 9a^2$ **16.** $2b^4 - 32$

Exercises 17–22: Solve the equation.

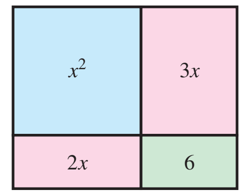
17. $x^2 - 16 = 0$ **18.** $y^2 = y + 20$

19. $9z^2 + 16 = 24z$ **20.** $x(x - 5) = 66$

21. $y^3 = 9y$ **22.** $x^4 - 5x^2 + 4 = 0$

23. A square has area $9x^2 + 30x + 25$. Find the length of a side in terms of x.

24. Write a polynomial in factored form that represents the total area of the rectangle.



25. ***Braking Distance*** The braking distance D in feet required for a car traveling at x miles per hour to stop on dry, level pavement can be modeled by $D = \frac{1}{11}x^2$.
(a) Calculate the distance required for the car to stop when it is traveling 55 miles per hour.
(b) If the braking distance is 99 feet, estimate the speed of the car.

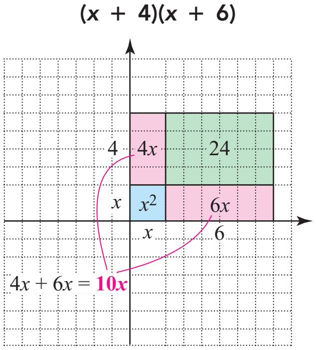
26. ***Flight of a Ball*** A ball is hit upward. Its height h in feet after t seconds is given by $h = -16t^2 + 48t + 4$. At what times is the ball 36 feet in the air?

CHAPTER 6 Extended and Discovery Exercises

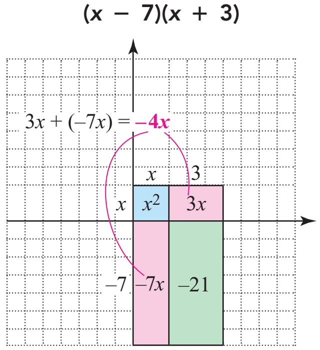
FACTORING TRINOMIALS VISUALLY

Exercises 1–6: A special grid similar to the xy-plane can be used to factor trinomials visually. The grid has four quadrants, where the area of any region located in quadrants I or III represents a positive term and the area of any region located in quadrants II or IV represents a negative term. The following factoring grids illustrate how to factor trinomials visually.

$x^2 + 10x + 24$ factors as $(x + 4)(x + 6)$



$x^2 - 4x - 21$ factors as $(x - 7)(x + 3)$



Factor the following trinomials visually.

1. $x^2 + 5x + 6$ **2.** $x^2 + 9x + 20$

3. $x^2 - 11x + 30$ **4.** $x^2 - 3x - 10$

5. $x^2 + 4x - 12$ **6.** $x^2 - 8x + 16$

THE DIFFERENCE OF TWO SQUARES

Exercises 7–12: ***Difference of Two Squares*** *The difference of two squares can be factored by using*

$$a^2 - b^2 = (a - b)(a + b).$$

This equation can also be used in some situations where an expression may not appear to be the difference of two squares. For example, because $(\sqrt{3})^2 = 3$, $x^2 - 3$ *can be written and then factored as*

$$x^2 - 3 = x^2 - (\sqrt{3})^2$$
$$= (x - \sqrt{3})(x + \sqrt{3}).$$

Use this concept to factor the following expressions as the difference of two squares.

7. $x^2 - 5$ **8.** $y^2 - 7$

9. $3z^2 - 25$ **10.** $7t^2 - 2$

11. $x - 4$ for $x \geq 0$ (*Hint:* $(\sqrt{x})^2 = x$.)

12. $x - 7$ for $x \geq 0$

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*Exercises 13–18: **Solving Equations** (Refer to Exercises 7–12.) Solve the equation by factoring it as the difference of two squares.*

13. $x^2 - 3 = 0$ **14.** $y^2 - 7 = 0$

15. $3x^2 - 25 = 0$ **16.** $7x^2 - 11 = 0$

17. $x^4 - 9 = 0$ **18.** $x^4 - 25 = 0$

CHAPTERS 1–6 Cumulative Review Exercises

Exercises 1 and 2: Evaluate by hand and then simplify to lowest terms.

1. $\frac{3}{5} \cdot \frac{15}{21}$ **2.** $\frac{4}{5} - \frac{1}{10}$

Exercises 3 and 4: Evaluate by hand.

3. $26 - 3 \cdot 6 \div 2$ **4.** $-2^2 + \frac{3 + 2}{8 + 2}$

5. Complete the table. Then use the table to solve the equation $2x + 3 = 5$.

x	-2	-1	0	1	2
$2x + 3$					

6. Translate the sentence "Triple a number decreased by 5 equals the number decreased by 7" into an equation using the variable n. Then solve the equation.

7. Convert 5.7% to fraction and decimal notation.

8. Solve $P = 2W + 2L$ for W.

9. Solve $5 - 3z < -1$.

10. Make a scatterplot having the following five points: $(-2, 3)$, $(-1, 2)$, $(0, -1)$, $(1, 1)$, and $(2, 2)$.

Exercises 11 and 12: Graph the given equation. Determine any intercepts.

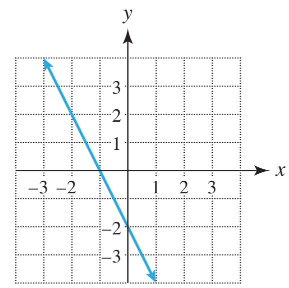
11. $y = 3x - 2$ **12.** $y = -2$

Exercises 13 and 14: Find the slope–intercept form for the line satisfying the given conditions.

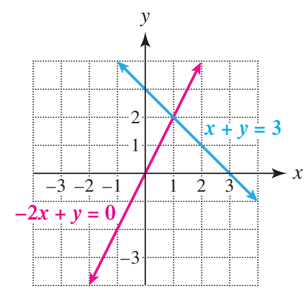
13. Perpendicular to $2x - 3y = -6$ and passing through the point $(1, 2)$

14. Passing through the points $(-2, 1)$ and $(1, 5)$

15. Identify the x-intercept and the y-intercept. Then write the slope–intercept form of the line.



16. The graphs of two equations are shown. Use the graphs to identify the solution to the system of equations. Then check your answer.



Exercises 17 and 18: Solve the system of equations.

17. $y = -1$
$2x + y = 1$

18. $5x + y = -5$
$-x + 2y = 12$

Exercises 19 and 20: Shade the solution set to the system of inequalities.

19. $x > -1$
$y < x$

20. $2x - y \leq 4$
$x + 2y \geq 2$

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Exercises 21–24: Simplify the expression.

21. -2^4

22. $(xy)^0$

23. $(xy)^4(x^3y^{-4})^2$

24. $7x^3(-2x^2 + 3x)$

Exercises 25–28: Simplify and write the expression using positive exponents.

25. $a^{-4} \cdot a^2$

26. $(2t^3)^{-2}$

27. $(xy)^{-3}(x^{-1}y^2)^{-1}$

28. $\left(\dfrac{2x}{y^{-2}}\right)^5$

Exercises 29 and 30: Divide and check.

29. $\dfrac{6x^3 + 12x^2}{3x}$

30. $\dfrac{3x^3 - x + 1}{x^2 + 1}$

Exercises 31–36: Factor completely.

31. $x^2 + 3x - 28$

32. $6y^2 + y - 12$

33. $25x^2 - 4y^2$

34. $64x^2 - 16x + 1$

35. $27t^3 - 8$

36. $-4x^2 + 4x + 24$

Exercises 37–40: Solve the equation.

37. $y^4 = 25y^2$

38. $8z^2 + 8z - 16 = 0$

39. $4z^3 = 49z$

40. $x^4 - 18x^2 + 81 = 0$

APPLICATIONS

41. ***Shoveling the Driveway*** Two people are shoveling snow from a driveway. The first person shovels 10 square feet per minute, while the second person shovels 8 square feet per minute.

(a) Write and simplify an expression that gives the total square feet that the two people shovel in x minutes.

(b) How many minutes would it take for them to clear a driveway with an area of 900 square feet?

42. ***Running Distance*** A person runs the following distances over three days: $1\frac{3}{4}$ miles, $2\frac{1}{2}$ miles, and $2\frac{2}{3}$ miles. How far does the person run altogether?

43. ***Burning Calories*** An athlete can burn 12 calories per minute while cross-country skiing and 9 calories per minute while running at 5 miles per hour. If the athlete burns 615 calories in 60 minutes, how long is spent on each activity?

44. ***Renting a Car*** A rental car costs \$20 plus \$0.25 per mile driven.

(a) Write an equation that gives the cost C of driving the car x miles.

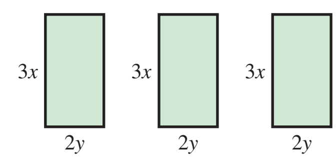
(b) Determine the number of miles that the car is driven if the rental cost is \$100.

45. ***Triangle*** In an isosceles triangle, the measures of the two smaller angles are equal and their sum is 20° more than the larger angle.

(a) Let x be the measure of one of the two smaller angles and y be the measure of the larger angle. Write a system of linear equations whose solution gives the measures of these angles.

(b) Solve your system.

46. ***Areas of Rectangles*** Find a monomial equal to the sum of the areas of the rectangles. Calculate this sum for $x = 2$ yards and $y = 3$ yards.



47. ***Area of a Square*** A square has area $x^2 + 12x + 36$. Find the length of a side.

48. ***Flight of a Golf Ball*** A golf ball is hit upward. Its height h in feet after t seconds is given by the formula $h = 64t - 16t^2$.

(a) How long does it take for the ball to hit the ground?

(b) At what times is the ball 48 feet in the air?

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