Sums and Differences vs Integrals and Derivatives

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This article offers one approach to the understanding, and also to the teaching, of the fundamental theorem of calculus. The ideas expressed here are not new—mathematically they cannot be original, and even pedagogically they might already be in use. If so, that is good. The author has become deeply involved in the effective presentation of calculus—in fact he has gone past the fatal point of no return, beyond which a book has to be written [1]. It is impossible to think about explaining this subject without searching for a fresh way to illuminate the relation of derivatives to integrals.

If the hopeful author of a calculus book is viewed with a lean and lively suspicion—as he certainly is—a small request might be made in return. You know this subject too well, and therefore you must forget what you know. Instead of the "suspension of disbelief" that is required of a playgoer, the reader is asked for something more difficult—a suspension of belief, and of a sure familiarity, in the ideas of calculus. All you are permitted is the knowledge that distance equals (constant) velocity multiplied by time. Of course you are encouraged to learn more.

In spite of the happy spirit in which this article is written, its goal is entirely serious. We all want mathematics to be appreciated. I think this only becomes possible when a part of the subject is understood. It might be unwise for us to transform every student into a full-scale mathematician, but there is no reason to fail in our more limited responsibility—to help students see some of the inspiration behind what they learn.

The Goal (in Two Parts) and the Plan

There are two goals, different but complementary. One is abstract, the other is very concrete. The first is to understand the relation of a function f to its derivative v, and the relation of v to its integral f. Starting cold (from the definitions) this is a substantial challenge. Working with graphs is better. Slope and area are tremendous visual supports. But my experience is that $v =$ velocity and $f =$ distance are the best. The intuition that comes from driving a car, and from ordinary use of the speedometer and odometer, is a free gift to calculus teachers. The relation of v to f is understood implicitly, and our contribution is to make it explicit.

The second goal is more special. The applications of calculus are built on a few functions (amazingly few). The student needs to become familiar with those particular functions. *They come in pairs f and v.* Where the first goal was to know the relation between them, the second goal is to know the functions themselves. Certain special pairs arise constantly in mathematical models, and we might as well connect with those v's and f's from the start.

The first pair is the one that everybody knows: *constant v and linear f*. (I don't skip them because they are known. I explain them because they are known!) The step between function and graph is not automatic in general, but that example is clear. Area and slope get a toehold—they are the area of a rectangle and the slope of a straight line. By no means are those ideas held in a firm grip. There is a long way to go, and the choice of the *next pair* is decisive.

A critical moment comes extremely early (often in the first week). Far ahead is the destination, to understand the key ideas and the key functions of calculus. We absolutely need a bridge. The goal cannot be reached from constant v-linear f in a single step. One approach is to start with *definitions* (of real numbers, functions, limits, ...) along with careful examples. I believe that is a seriously wrong start. The approach is taken in good faith, but I am convinced there is a better one. I will concentrate on explaining how a calculus course can go forward from the first pair, to build on the intuition that made it understood before it was expressed.

The second pair is *one step away* from the first pair. It is a piecewise constant v and a piecewise linear f—the best motion seems to be *forward and back*. The velocity is $+V$ up to time T, and then $-V$ from T to $2T$. Every student knows what this motion does, and where it ends up. The distance function f is turned into a graph (Figure 1). The velocity is also expressed by a graph (and negative area for negative v makes sense). The area under the v-graph should be checked at $T/2$ and $3T/2$. It is essential to take time with visual understanding (including the slope) because the next step will grow out of this one.

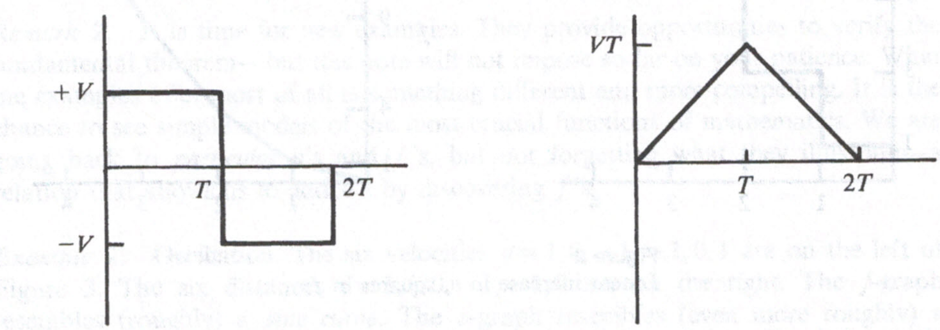

Figure 1
Forward and back: the v-graph and the f-graph.

Now I can describe the plan. *It is to imitate calculus by algebra*. The velocity-distance relation can be made plain when v changes by finite steps. There is a sequence of velocities $v_1, v_2, v_3, \ldots$. For convenience the jumps occur at evenly spaced times $t = 1, 2, 3, \ldots$. The distances reached at those times are $f_1, f_2, f_3, \ldots$. We are dealing with functions, and at the same time with numbers. A typical case is

$$v = 1, 2, 3, 4, \ldots \quad \text{and} \quad f = 1, 3, 6, 10, \ldots.$$

The one thing every student can do is to fill in the next v and f! The pattern is seen; but it has to be written down. Mathematics is about *discovering* patterns and then *describing* them—and the way to teach mathematics is to do it.

Calculus Before Limits

I will propose four specific v's and f's. They are discrete models of familiar functions. The velocity changes, not continuously but in steps. Calculus will deal with continuous change—that is its central idea—and these models simulate that change by a sequence of jumps.

1. A steadily increasing velocity: $v = 1, 3, 5, 7, \ldots$ imitates $v = 2t$.
2. An oscillating velocity: $v = 1, 0, -1, -1, 0, 1, \ldots$ imitates $v = \cos t$.
3. An exponentially increasing velocity: $v = 1, 2, 4, 8, \ldots$ imitates $v = 2^t$.
4. A burst of speed (then stop): $v = 100, 0, 0, \ldots$ makes f imitate a step function.

The oscillation could come ahead of the others, because it is closest to forward-back motion and has the most attractive graph. But the first case of continuous change, when calculus really starts, will be $v = 2t$. Its discrete form yields a neat pattern—comparable to $v = 1, 2, 3, 4$ but with better f's. The presence of squares is concealed by $f = 1, 3, 6, 10$—now it will stand out.

Example 1. **Increasing velocity.** The four velocities 1, 3, 5, 7 are displayed on the left of Figure 2. The distance function is on the right. The goal is to uncover, numerically and then algebraically, the relation between

$$v = 1, 3, 5, 7, \ldots \quad \text{and} \quad f = 1, 4, 9, 16, \ldots.$$

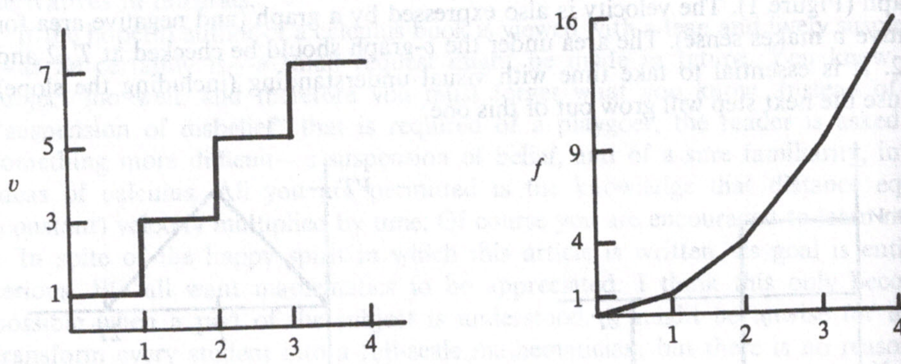

Figure 2
Linear increase in v, squares in f.

At $t = 1$ the distance traveled (starting from zero) is $f_1 = 1$. It equals the velocity (which is 1) multiplied by the time (also 1). At $t = 2$ the distance is $f_2 = 1 + 3$—the velocity in the second time interval is $v_2 = 3$. After three steps f_3 is $v_1 + v_2 + v_3 = 1 + 3 + 5 = 9$. It is a pleasure to prove that the f's are perfect squares, but more important is their relation to the v's.

The first step begins with the f's. We are given 1, 4, 9, 16, 25 and we are aiming for 1, 3, 5, 7, 9. Those v's are the *differences* of the f's:

$$v_j \textit{ is the difference } f_j - f_{j-1}. \tag{1}$$

The velocity $v_5 = 9$ is the difference $25 - 16$. It is the slope of the f-graph in the fifth interval. Similarly $v_4 = 7$ is $16 - 9$. I admit to a small difficulty in the first interval (at $j = 1$), from the fact there is no f_0. For this example the natural choice is $f_0 = 0$. The need for a starting point will come back to haunt us (or help us) in calculus.

Remark 1. The equation $v_j = f_j - f_{j-1}$ asks the student to take a step that we no longer notice. With numbers everything is clear—but now we have switched to letters. There is a risk of muddying up the whole thing by algebra. But we have to do it. The investment in learning algebra is partly justified by our use of it here.

Now comes the crucial step, in the *reverse direction*. How do we recover the f's from the v's? The whole relation between differential calculus and integral calculus rests on the fact that **taking sums is the inverse of taking differences.** We saw the discrete form of differentiation in equation (1), and we will see the discrete form of integration in equation (2).

To repeat: The numbers 1, 3, 5, 7, 9 are given. The numbers 1, 4, 9, 16, 25 are wanted. The student sees how to do it, but the professor sees more. This numerical example carries within it—in a limited but absolutely convincing form—the fundamental theorem of calculus. It is still algebra, because v is piecewise constant—but the relation between v and f is the real thing. Why not present it now?

Fundamental Theorem of Calculus (before limits). *If each* $v_j = f_j - f_{j-1}$ *then*

$$\text{the sum } v_1 + v_2 + \cdots + v_n \text{ equals } f_n - f_0. \tag{2}$$

The area under the graph of v is the change in f.

Proof. We are adding $(f_1 - f_0) + (f_2 - f_1) + (f_3 - f_2) + \cdots + (f_n - f_{n-1})$. The first f_1 is canceled by $-f_1$. Then f_2 is canceled by $-f_2$. At the end only $-f_0$ and $+f_n$ are left. In case $f_0 = 0$, the sum of v's has telescoped into f_n. This is the distance traveled, or the area under v.

Remark 2. It is time for new examples. They provide opportunities to verify the fundamental theorem—but this note will not impose so far on your patience. What the examples offer most of all is something different and more compelling. It is the chance to see simple models of the most crucial functions of mathematics. We are going back to *particular* v's and f's, but not forgetting what they illustrate—a relation that allows us to add v's by discovering f's.

Example 2. **Oscillation**. The six velocities $v = 1, 0, -1, -1, 0, 1$ are on the left of Figure 3. The six distances $f = 1, 1, 0, -1, -1, 0$ are on the right. The f-graph resembles (roughly) a *sine curve*. The v-graph resembles (even more roughly) a *cosine curve*. The f's were formed by the same rule as before, that each f_j equals the sum $v_1 + \cdots + v_j$. It is the total distance, when the velocities are held for one time unit each.

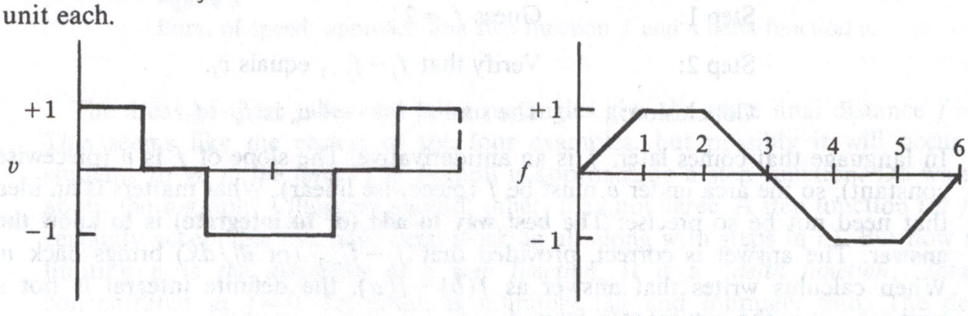

Figure 3
Piecewise constant cosine and piecewise linear sine.

These discrete sines and cosines share three properties (possibly more) with the true waveforms $\sin t$ and $\cos t$:

1. The slope of the discrete sine is the discrete cosine, and the area under the discrete cosine is the discrete sine. (Note: This is true for all v-f pairs—it is not particular to this one.)

2. The f_j are one time step behind the v_j. In other words $v_j = f_{j+1}$—the new distance is the old velocity. Since v_j is always $f_j - f_{j-1}$, that cancels f_{j+1} on the left side of a difference equation:

$$(f_{j+1} - f_j) - (f_j - f_{j-1}) = f_{j+1} - 2f_j + f_{j-1} = -f_j. \tag{3}$$

This imitates an important differential equation (second derivative of $\sin t$ equals $-\sin t$). The same is true of the cosines v_j. The time lag $f_{j+1} = v_j$ is a close copy of $\sin(t + \pi/2) = \cos t$—but the delay is $1/6$ of the period instead of $1/4$.

3. The motion returns to the start at $t = 6$ because $v_1 + v_2 + \cdots + v_6 = 0$. The numbers v and f could repeat over the next six intervals, and the graphs would be exactly the same. These are *periodic* waveforms. (*Note*: It would be interesting to take equation (3) as fundamental, and show that the period must be 6.)

The true sine and cosine possess other properties that f and v do not imitate. The outstanding example is $(\cos t)^2 + (\sin t)^2 = 1$. If desired, that could be recovered by the faster oscillation $v = 1, -1, -1, 1$ and $f = 1, 0, -1, 0$ (repeated periodically). However we have to choose the *averages* of v at the jumps, and work with $V_1 = 0$, $V_2 = -1$, $V_3 = 0$, $V_4 = 1$. Then $V_j^2 + f_j^2 = 1$. In many ways the averages are more reasonable anyway.

Example 3. **Exponential increase** (powers of 2). In this example v doubles at every step: $v = 1, 2, 4, 8, 16, \ldots$ The velocity in the jth time interval is $v = 2^{j-1}$. The novelty comes for the distance f, *which does not begin at zero*. Its starting value must be $f_0 = 2^0 = 1$, if f is to imitate the exponential. Then the next distances are

$$f_1 = 1 + 1 = 2, \qquad f_2 = 1 + 1 + 2 = 4, \qquad f_3 = 1 + 1 + 2 + 4 = 8.$$

The formula is $f_j = 2^j$. It jumps out from Figure 4. On a normal day students won't want a proof. So don't call it a proof! It is a beautiful chance to use the fundamental theorem $\Sigma(\Delta f) = f_n - f_0$ as a way to add up the v's by guessing the

Step 1:	Guess $f_j = 2^j$.
Step 2:	Verify that $f_j - f_{j-1}$ equals v_j.
Conclusion:	The sum $v_1 + \cdots + v_n$ is $f_n - f_0$.

In language that comes later, f is an antiderivative. The slope of f is v (piecewise constant), so the area under v must be f (piecewise linear). What matters is an idea that need not be so precise: **The best way to add (or to integrate) is to know the answer.** The answer is correct, provided that $f_j - f_{j-1}$ (or df/dx) brings back v. When calculus writes that answer as $f(b) - f(a)$, the definite integral is not a surprise—it is the natural step from $f_n - f_0$.

The exercise of actually summing the powers of 2 is not to be missed.

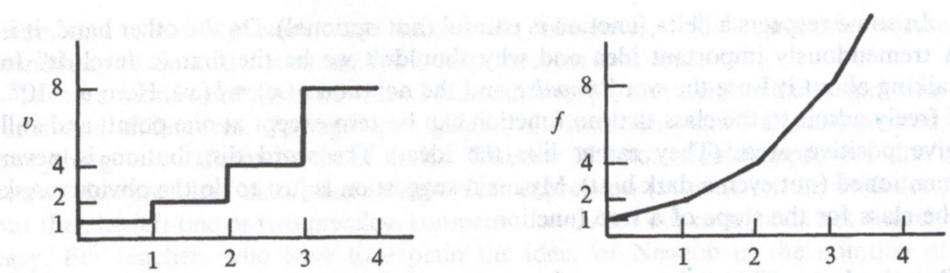

Figure 4
Exponentials $v_j = 2^{j-1}$ and $f_j = 2^j$.

Note. The twin example has exponential *decrease*. The velocities $v = -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \ldots$ are negative. The distances $f = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ are positive (because $f_0 = 1$). That example gives 2^{-j} a meaning beyond pure manipulation of exponents.

Example 4. **A short burst of speed.** This last example gives a proper importance to the dimension of time. With unit time intervals the slopes were $f_j - f_{j-1}$, and no student is going to notice the division by one. A check on dimensions is the simplest test of any formula, from $f = vt$ to $e = mc^2$, and here are four v's:

$v = 10$	up to	$t = \frac{1}{10}$	(then stop)	$v = 100$	up to	$t = \frac{1}{100}$	(then stop)
$v =$ ____	up to	$t = \frac{1}{10{,}000}$	(stop)	$v = 10^n$	up to	$t =$ ____	(stop)

Exercise 1 is to fill in the blanks. Exercise 2 is to draw the graphs of f (with $f_0 = 0$). Exercise 3 (the mathematical one) is to say what those f-graphs have in common, and then *what the v-graphs have in common* (equal areas).

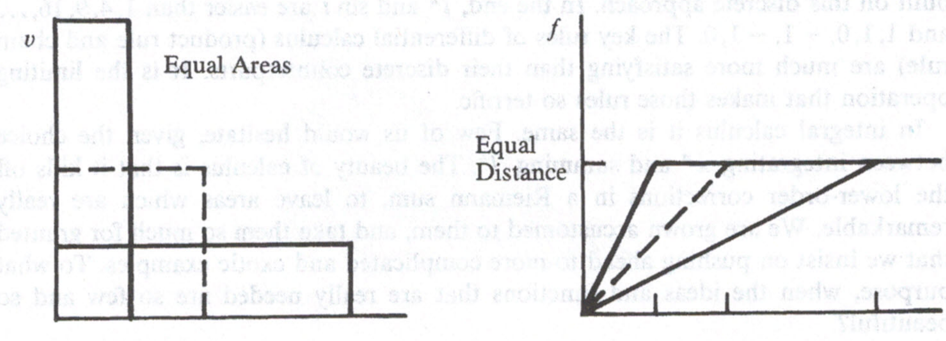

Figure 5
Burst of speed: approach to a step function f and a delta function v.

The areas of these taller and taller rectangles give the same final distance $f = 1$. This seems like the easiest of the four examples, but possibly it will occur to students to *go to the limit*. The f-graph is approaching a step function. Do we ask about the v-graph? I think we should; others may not agree. A step function for f is certainly acceptable (we have been working all along with steps in v). But now the limiting v is *the derivative of a step function*. It is a "*delta function*," totally concentrated at $t = 0$. Its graph is infinitely tall and infinitely thin. The delta function is zero except at one point, but its integral is the jump in f.

In some respects a delta function is painful (but optional). On the other hand, it is a tremendously important idea and why shouldn't we be the first to teach it? In talking about it I use the word *impulse* and the notation $v(x) = \delta(x)$. Here $v = 10\delta$. I freely admit to the class that no function can be zero except at one point, and still give positive area. (They rather like the idea.) The word distribution is never mentioned (not even a dark hint). My main suggestion is just to do the obvious: Ask the class for the slope of a step function.

Conclusion. The author owes the reader a brief accounting. What is achieved, and what is left? Our purpose was to prepare for one of the central ideas of calculus—the relation between v and f. It is not the *fact* of the fundamental theorem, but the *meaning of that fact* and the *application of that fact*, that our teaching has to aim for.

We proposed two goals. One was general (the v-f relation), the other was particular (special v-f pairs). The examples were limited to piecewise constant velocities, but that limitation brought a reward—the functions are accessible to students at the very start of a course. They give something valuable to do (and new functions to create) at that early time when so few tools are available. Psychologically this time is crucial. Like the rest of us, students form quick opinions. If the course can't get itself started, and the ideas are drowned in an ocean of definitions, those opinions will not be what we hope for.

For college freshmen who studied calculus in high school, the first days are decisive. If nothing is new they stop listening. The models suggested here are likely to be new—and the discrete-continuous analogy deserves all the emphasis we can give it.

A note of caution. I am absolutely not proposing that the whole calculus course be built on this discrete approach. In the end, t^2 and $\sin t$ are easier than $1, 4, 9, 16, \ldots$ and $1, 1, 0, -1, -1, 0$. The key rules of differential calculus (product rule and chain rule) are much more satisfying than their discrete counterparts. It is the limiting operation that makes those rules so terrific.

In integral calculus it is the same. Few of us would hesitate, given the choice between integrating x^p and summing j^p. The beauty of calculus is that it kills off the lower-order corrections in a Riemann sum, to leave areas which are really remarkable. We are grown accustomed to them, and take them so much for granted that we insist on pushing ahead to more complicated and exotic examples. To what purpose, when the ideas and functions that are really needed are so few and so beautiful?

I want to emphasize that the new textbook **[1]** is aimed at careful change, not revolution. The work of calculus is still there to do, but it need not be drudgery. My effort is to make the book lighter, spiritually as well as materially. I hope others are also writing, to make progress on the problems that the 1987 colloquium presented so forcefully **[2]**.

This short article may bring to mind other ideas and other functions. The velocities $1, r, r^2, r^3, \ldots$ connect f to the sum of a geometric series. But "calculus before limits" doesn't last long. The derivative is introduced very soon, and (separately) the integral appears. Functions are minimized and $y' = cy$ is solved. Still the ideas described here do come back for one short and vital moment. Slope and area have to be brought together into the fundamental theorem of calculus. It is there that the early work with sums and differences, which gave the course something to do at the start, makes a deep theorem look true.

It was a moment of pleasure when I learned from **[3]** that Leibniz had begun with sums and differences. Perhaps they guided his intuition—until Cauchy arrived, everything was calculus without limits. Whether Newton would approve I don't know. He cared less than others about clear expression (at least on the evidence), but then he left one or two priceless compensations. And our own job is not entirely easy. For teachers who have to explain the ideas of Newton in the notation of Leibniz to pupils who may not be quite so apt as Cauchy, every source of illumination is welcome.

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References

1. Gilbert Strang, *Calculus*, Wellesley-Cambridge Press, Wellesley, MA (to be published in 1990).
2. Lynn Steen, ed., *Calculus for a New Century*, Mathematical Association of America, Washington, 1988.
3. C. H. Edwards, *The Historical Development of the Calculus*, Springer Verlag, New York, 1979.