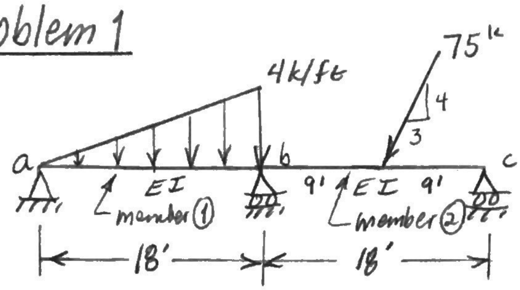
Problem 1



EI constant

1) Problem Details: Dof's are θ_a, θ_b, & θ_c. Since only rotations at supports are the unknown's, then we can use 2X2 rotational stiffness matrix

2) Calculate the fixed-end force vector for each member, $\{f^F\}_m$

Member 1: $FEM_{ab} = -\dfrac{wL^2}{30} = -\dfrac{4(18)^2}{30} = -43.2 \text{ k-ft}$

$FEM_{ba} = +\dfrac{wL^2}{20} = \dfrac{4(18)^2}{20} = +64.8 \text{ k-ft} \Rightarrow \{f^F\}_1 = \begin{Bmatrix} -43.2 \\ 64.8 \end{Bmatrix} \begin{matrix} \theta_a \\ \theta_b \end{matrix}$

Member 2: $FEM_{bc} = -\dfrac{PL}{8} = -\dfrac{75(\frac{4}{5})(18)}{8} = -135 \text{ k-ft}$

$FEM_{cb} = +\dfrac{PL}{8} = +135 \text{ k-ft} \Longrightarrow \{f^F\}_2 = \begin{Bmatrix} -135 \\ 135 \end{Bmatrix} \begin{matrix} \theta_b \\ \theta_c \end{matrix}$

3) Calculate member stiffness $[\bar{k}]$ for each member

Member 1: a-b, EI, L=18'

$$[\bar{k}]_{ab} = \frac{2EI}{L}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{2EI}{18}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = EI \begin{array}{cc} \theta_a & \theta_b \\ \left[\begin{matrix} 2/9 & 1/9 \\ 1/9 & 2/9 \end{matrix}\right] \end{array} \begin{matrix} \theta_a \\ \theta_b \end{matrix}$$

Member 2: b-c, EI, L=18'

$$[\bar{k}]_{bc} = \frac{2EI}{18}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = EI \begin{array}{cc} \theta_b & \theta_c \\ \left[\begin{matrix} 2/9 & 1/9 \\ 1/9 & 2/9 \end{matrix}\right] \end{array} \begin{matrix} \theta_b \\ \theta_c \end{matrix}$$

4) Assemble global structure stiffness matrix [K] & nodal force vector

$$[K] = EI \begin{array}{ccc} \theta_a & \theta_b & \theta_c \\ \left[\begin{matrix} 2/9 & 1/9 & 0 \\ 1/9 & (2/9+2/9) & 1/9 \\ 0 & 1/9 & 2/9 \end{matrix}\right] \end{array} \begin{matrix} \theta_a \\ \theta_b \\ \theta_c \end{matrix} = EI \begin{bmatrix} 2/9 & 1/9 & 0 \\ 1/9 & 4/9 & 1/9 \\ 0 & 1/9 & 2/9 \end{bmatrix}$$

$$\{F\} = -\{F^F\} = -\begin{Bmatrix} -43.2 \\ 64.8+(-135) \\ 135 \end{Bmatrix} = \begin{Bmatrix} 43.2 \\ 70.2 \\ -135 \end{Bmatrix} \text{k-ft} \quad \begin{matrix} \theta_a \\ \theta_b \\ \theta_c \end{matrix}$$

5) Solve $[K]\{\Delta\}=\{F\}$

$$EI\begin{bmatrix} 2/9 & 1/9 & 0 \\ 1/9 & 4/9 & 1/9 \\ 0 & 1/9 & 2/9 \end{bmatrix}\begin{Bmatrix}\theta_a\\ \theta_b\\ \theta_c\end{Bmatrix}=\begin{Bmatrix}43.2\\ 70.2\\ -135\end{Bmatrix} \Rightarrow \begin{Bmatrix}\theta_a\\ \theta_b\\ \theta_c\end{Bmatrix}=\frac{1}{EI}\begin{Bmatrix}20.25\\ 348.3\\ -781.65\end{Bmatrix}$$

k-ft²

6) Compute member forces, $\{f\}_m=[\bar{k}]_m\{\Delta\}_m+\{f^F\}_m$

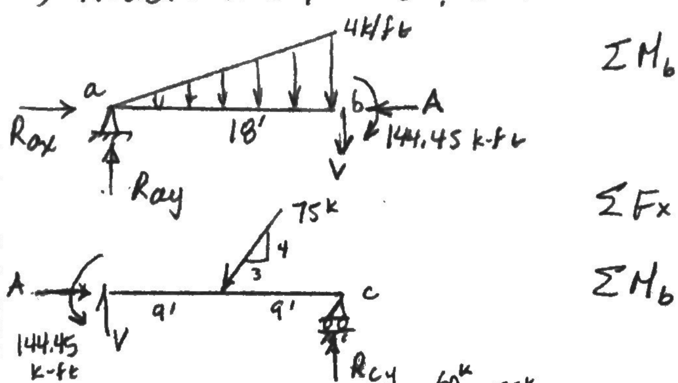
Member 1: $$\{f_{ab}\}=EI\begin{bmatrix}2/9 & 1/9\\ 1/9 & 2/9\end{bmatrix}\frac{1}{EI}\begin{Bmatrix}20.25\\ 348.3\end{Bmatrix}+\begin{Bmatrix}-43.2\\ 64.8\end{Bmatrix}$$

$$=\begin{Bmatrix}43.2\\ 79.65\end{Bmatrix}+\begin{Bmatrix}-43.2\\ 64.8\end{Bmatrix}=\begin{Bmatrix}0\\ 144.45\end{Bmatrix}\text{ k-ft}$$

Member 2: $$\{f_{bc}\}=EI\begin{bmatrix}2/9 & 1/9\\ 1/9 & 2/9\end{bmatrix}\frac{1}{EI}\begin{Bmatrix}348.3\\ -781.65\end{Bmatrix}+\begin{Bmatrix}-135\\ 135\end{Bmatrix}$$

$$=\begin{Bmatrix}-9.45\\ -135\end{Bmatrix}+\begin{Bmatrix}-135\\ 135\end{Bmatrix}=\begin{Bmatrix}-144.45\\ 0\end{Bmatrix}\text{ k-ft}$$

7) Reactions & SFD & BMD:



$\Sigma M_b=0$: $144.45-\frac{1}{2}(4)(18)\frac{1}{3}(18)+18R_{ay}=0$

$\Rightarrow R_{ay}=+3.975^k\uparrow$

$\Sigma F_x=0 \xrightarrow{+}$: $R_{ax}-A=0 \Rightarrow R_{ax}=A$

$\Sigma M_b=0$: $-144.45-18R_{cy}+\frac{4}{5}(75)(9)=0$

$\Rightarrow R_{cy}=21.975^k\uparrow$

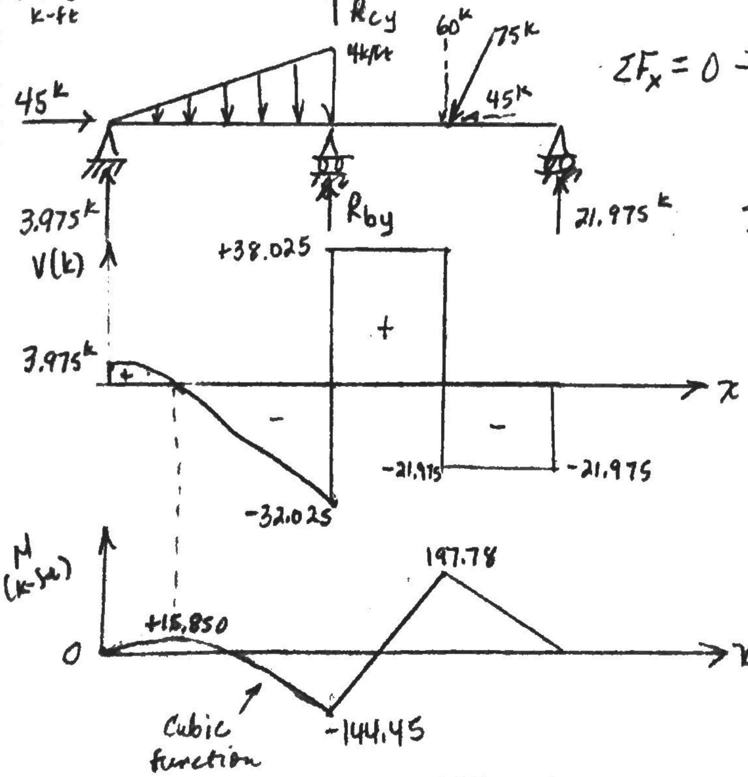
$\Sigma F_x=0 \xrightarrow{+}$: $A-\frac{3}{5}(75)=0 \Rightarrow A=45^k$

so $R_{ax}=45^k\rightarrow$

$\Sigma F_y=0\ \uparrow+$ (whole beam)

$3.975-\frac{1}{2}(4)(18)+R_{by}-60+21.975=0$

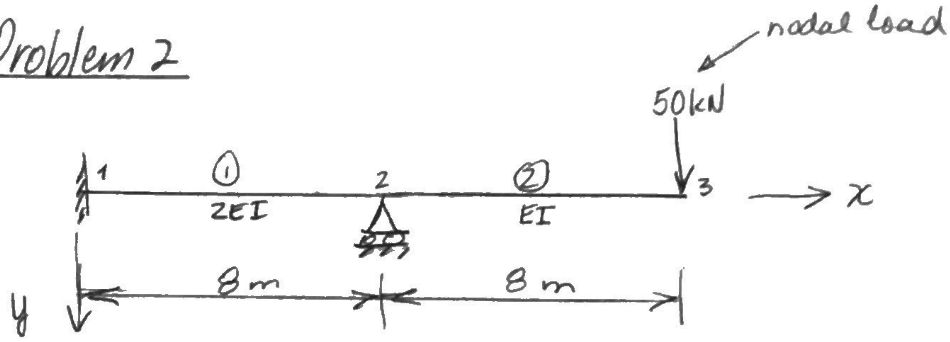
$\Rightarrow R_{by}=70.05^k\uparrow$



$V_{x<18}=-\frac{x^2}{9}+3.975$

$M_{x<18}=-\frac{x^3}{27}+3.975x$

Problem 2



1) Problem Details : - joints & members numbered & global axes defined above. DoF's : θ_2 , θ_3 & Δ_3
Restrained displ. θ_1 , Δ_1 & Δ_2

2) Calculate fixed-end force vector for each member, $\{f^F\}_m$
No transverse loading between the joints so $\{f^F\}_1 = \{f^F\}_2 = 0$

3) Calculate member stiffness matrix $[k]$ for each member:

$$[k] = \frac{EI}{L^3}\begin{bmatrix} 4L^2 & 2L^2 & 6L & -6L \\ 2L^2 & 4L^2 & 6L & -6L \\ 6L & 6L & 12 & -12 \\ -6L & -6L & -12 & 12 \end{bmatrix} = EI\begin{bmatrix} 4/L & 2/L & 6/L^2 & -6/L^2 \\ & 4/L & 6/L^2 & -6/L^2 \\ & & 12/L^3 & -12/L^3 \\ \text{Symm.} & & & 12/L^3 \end{bmatrix}$$

Member ① : 1-2 , 2EI & L = 8m

$$[k]_1 = [k']_1 = EI \begin{array}{c} \begin{array}{cccc} \theta_1 & \theta_2 & \Delta_1 & \Delta_2 \end{array} \\ \begin{bmatrix} 1 & 0.5 & 0.1875 & -0.1875 \\ 0.5 & 1 & 0.1875 & -0.1875 \\ .1875 & .1875 & 0.046875 & -0.046875 \\ -.1875 & -.1875 & -.046875 & 0.046875 \end{bmatrix} \end{array} \begin{array}{c} \theta_1 \\ \theta_2 \\ \Delta_1 \\ \Delta_2 \end{array}$$

(brought factor of 2 inside matrix)

Member ② : 2-3 , EI & L = 8m

$$[k]_2 = [k']_2 = EI \begin{array}{c} \begin{array}{cccc} \theta_2 & \theta_3 & \Delta_2 & \Delta_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.25 & .09375 & -.09375 \\ 0.25 & 0.5 & .09375 & -.09375 \\ \cancel{.09375} & \cancel{.09375} & \cancel{.0234375} & \cancel{-.0234375} \\ -.09375 & -.09375 & -.0234375 & .0234375 \end{bmatrix} \end{array} \begin{array}{c} \theta_2 \\ \theta_3 \\ \Delta_2 \\ \Delta_3 \end{array}$$

4) Assemble structure's global stiffness matrix $[K]$ & nodal force vector $\{F\}$ (reduced set of eqs.)

$$[K] = EI \begin{array}{c} \begin{array}{ccc} \theta_2 & \theta_3 & \Delta_3 \end{array} \\ \begin{bmatrix} (1+0.5) & 0.25 & -.09375 \\ 0.25 & 0.5 & -.09375 \\ -.09375 & -.09375 & .0234375 \end{bmatrix} \end{array} \begin{array}{c} \theta_2 \\ \theta_3 \\ \Delta_3 \end{array}$$

$$\{F\} = \begin{Bmatrix} 0 \\ 0 \\ 50\text{ kN} \end{Bmatrix} \begin{array}{c} \theta_2 \\ \theta_3 \\ \Delta_3 \end{array}$$

(+y-axis pointing down)

5) Solve $\{F\} = [K]\{\Delta\}$

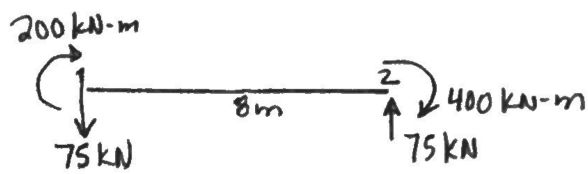
$$EI \begin{bmatrix} 1.5 & 0.25 & -.09375 \\ 0.25 & 0.5 & -.09375 \\ -.09375 & -.09375 & .0234375 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \\ \Delta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \theta_2 \\ \theta_3 \\ \Delta_3 \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 400 \\ 2000 \\ 11733.33 \end{Bmatrix} \begin{matrix} \circlearrowright \\ \circlearrowright \\ m \downarrow \end{matrix}$$

6) Compute member forces $\{f\}_m = [k]_m \{\Delta\}_m + \{f^F\}_m$

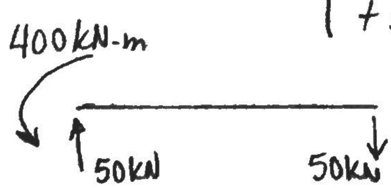
Member ①: $\{f\}_{1-2} = \begin{Bmatrix} M_1 \\ M_2 \\ V_1 \\ V_2 \end{Bmatrix} = EI \begin{bmatrix} 1 & 0.5 & .1875 & -.1875 \\ & 1 & .1875 & -.1875 \\ \text{symm.} & & .046875 & -.046875 \\ & & & .046875 \end{bmatrix} \frac{1}{EI} \begin{Bmatrix} 0 \\ 400 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

$$= \begin{Bmatrix} 200 \text{ kN-m } \circlearrowright \\ 400 \text{ kN-m } \circlearrowright \\ 75 \text{ kN } \downarrow \\ -75 \text{ kN } \uparrow \end{Bmatrix}$$



Member ②: $\{f\}_{2-3} = \begin{Bmatrix} M_2 \\ M_3 \\ V_2 \\ V_3 \end{Bmatrix} = EI \begin{bmatrix} 0.5 & 0.25 & .09375 & -.09375 \\ & 0.5 & .09375 & -.09375 \\ \text{Symm.} & & .0234375 & -.0234375 \\ & & & .0234375 \end{bmatrix} \frac{1}{EI} \begin{Bmatrix} 400 \\ 2000 \\ 0 \\ 11733.33 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

$$= \begin{Bmatrix} -400 \text{ kN-m } \circlearrowleft \\ 0 \\ -50 \text{ kN } \uparrow \\ +50 \text{ kN } \downarrow \end{Bmatrix}$$



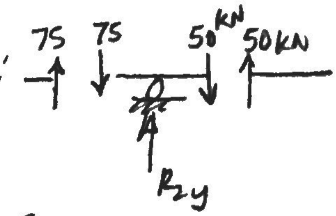
7) Reactions:

$M_1 = 200 \text{ kN-m } \circlearrowright$

$R_{1y} = 75 \text{ kN } \downarrow$

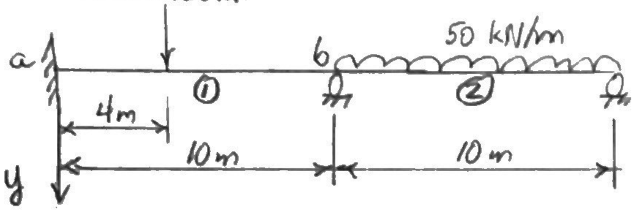
$R_{1x} = 0 \text{ kN}$

R_{2y}:



$\Sigma F_y = 0: \; R_{2y} - 75 - 50 = 0 \Rightarrow R_{2y} = 125 \text{ kN} \uparrow$

Problem 3



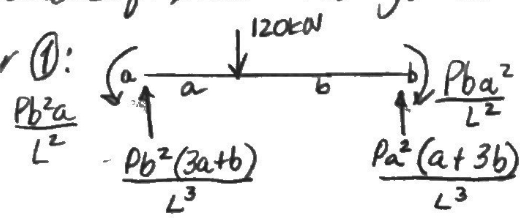
EI = constant = 400 MN-m^2
= 400 x 10^3 kN-m^2
"b" settles 0.03 m ↓

Solution 1 – use 4x4 stiffness matrix & account for settlement in global stiffness eq. $\{F\} = [K]\{\Delta\}$

1) Problem details: – number joints & members & define global axes
- DOF's: θ_2 & θ_3
- Prescribed displ.: $\Delta_b = 0.03$ m ↓ (+ according to defined global axes)
- Restrained displ.: $\theta_a = 0$, $\Delta_a = 0$, $\Delta_c = 0$

2) Calculate fixed-end force vector for each member, $\{f^F\}_m$

Member ①:



$M_a^F = -\dfrac{Pb^2a}{L^2} = -\dfrac{(120)(6)^2(4)}{10^2} = -172.8$ kN-m

$M_b^F = +\dfrac{Pba^2}{L^2} = \dfrac{120(6)(4)^2}{10^2} = +115.2$ kN-m

$V_a^F = -\dfrac{Pb^2(3a+b)}{L^3} = -\dfrac{120(6)^2(3(4)+6)}{10^3}$

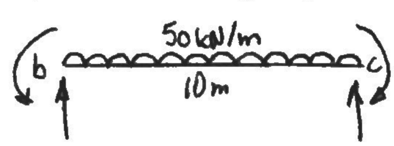
$= -77.76$ kN
↑ according to defined axes

$V_b^F = -\dfrac{Pa^2(a+3b)}{L^3} = -\dfrac{120(4)^2(4+3(6))}{10^3}$

$= -42.24$ kN

$$\{f^F\}_1 = \begin{Bmatrix} -172.8 \text{ kN-m} \\ 115.2 \text{ kN-m} \\ -77.76 \text{ kN} \\ -42.24 \text{ kN} \end{Bmatrix} \begin{matrix} \theta_a \\ \theta_b \\ \Delta_a \\ \Delta_b \end{matrix}$$

Member ②:



$M_b^F = -\dfrac{wL^2}{12} = \dfrac{-50(10)^2}{12} = -416.67$ kN-m

$M_c^F = +416.67$ kN-m

$V_b^F = -\dfrac{wL}{2} = -\dfrac{50(10)}{2} = -250$ kN

$V_c^F = -250$ kN

$$\{f^F\}_2 = \begin{Bmatrix} -416.67 \text{ kN-m} \\ 416.67 \text{ kN-m} \\ -250 \\ -250 \end{Bmatrix} \begin{matrix} \theta_b \\ \theta_c \\ \Delta_b \\ \Delta_c \end{matrix}$$

3) Calculate member stiffness matrix $[k]$ for each member

Member ①: a-b, $L = 10\,m$, $EI = 400 \times 10^3\ kN\text{-}m^2$

$$[k]_1 = \frac{2EI}{10}\begin{bmatrix} 2 & 1 & 0.3 & -0.3 \\ 1 & 2 & 0.3 & -0.3 \\ 0.3 & 0.3 & 0.06 & -0.06 \\ -0.3 & -0.3 & -0.06 & 0.06 \end{bmatrix} = EI \begin{array}{c} \begin{array}{cccc} \theta_a & \theta_b & \Delta_a & \Delta_b \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.06 & -0.06 \\ 0.2 & 0.4 & 0.06 & -0.06 \\ 0.06 & 0.06 & 0.012 & -0.012 \\ -0.06 & -0.06 & -0.012 & 0.012 \end{bmatrix} \end{array} \begin{array}{c} \theta_a \\ \theta_b \\ \Delta_a \\ \Delta_b \end{array}$$

$$[k]_2 = EI \begin{array}{c} \begin{array}{cccc} \theta_b & \theta_c & \Delta_b & \Delta_c \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.06 & -0.06 \\ 0.2 & 0.4 & 0.06 & -0.06 \\ 0.06 & 0.06 & 0.012 & -0.012 \\ -0.06 & -0.06 & -0.012 & 0.012 \end{bmatrix} \end{array} \begin{array}{c} \theta_b \\ \theta_c \\ \Delta_b \\ \Delta_c \end{array}$$

4) Assemble global structure stiffness matrix $[K]$ & nodal force vector $\{F\}$ $(\Delta_b = 0.03\,m)$

$$\{F\} = \begin{array}{c} \theta_b \\ \theta_c \\ \theta_a \\ \Delta_a \\ \Delta_b \\ \Delta_c \end{array} \begin{Bmatrix} 0 \\ 0 \\ M_a \\ R_{ay} \\ R_{by} \\ R_{cy} \end{Bmatrix} - \begin{Bmatrix} 115.2 + (-416.67) \\ 416.67 \\ -172.8 \\ -77.76 \\ -42.24 + (-250) \\ -250 \end{Bmatrix} = \begin{Bmatrix} 301.47 \\ -416.67 \\ M_a + 172.8 \\ R_{ay} + 77.76 \\ R_{by} + 292.24 \\ R_{cy} + 250 \end{Bmatrix}$$

$$\begin{Bmatrix} 301.47 \\ -416.67 \\ M_a + 172.8 \\ R_{ay} + 77.76 \\ R_{by} + 292.24 \\ R_{cy} + 250 \end{Bmatrix} = EI \begin{array}{c} \begin{array}{cccccc} \theta_b & \theta_c & \theta_a & \Delta_a & \Delta_b & \Delta_c \end{array} \\ \left[\begin{array}{cc|cccc} (0.4+0.4) & 0.2 & 0.2 & 0.06 & (-0.06+0.06) & -0.06 \\ 0.2 & 0.4 & 0 & 0 & 0.06 & -0.06 \\ \hline 0.2 & 0 & 0.4 & 0.06 & -0.06 & 0 \\ 0.06 & 0 & 0.06 & 0.012 & -0.012 & 0 \\ (-0.06+0.06) & 0.06 & -0.06 & -0.012 & (0.012+0.012) & -0.012 \\ -0.06 & -0.06 & 0 & 0 & -0.012 & 0.012 \end{array}\right] \end{array} \begin{Bmatrix} \theta_b \\ \theta_c \\ \theta_a = 0 \\ \Delta_a = 0 \\ \Delta_b = 0.03 \\ \Delta_c = 0 \end{Bmatrix}$$

5) Use partitioned matrices eqs. to account for settlement i.e. $\Delta_s \neq 0$ and solve for unknown displ.

$$\{Q_f\} = [K_{11}]\{\Delta_f\} + [K_{12}]\{\Delta_s\}$$

$$\begin{Bmatrix} 301.47 \\ -416.67 \end{Bmatrix} = EI \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} \begin{Bmatrix} \theta_b \\ \theta_c \end{Bmatrix} + EI \begin{bmatrix} 0.2 & 0.06 & 0 & -0.06 \\ 0 & 0 & 0.06 & -0.06 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.03 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 301.47 \\ -416.67 \end{Bmatrix} = 400 \times 10^3 \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} \begin{Bmatrix} \theta_b \\ \theta_c \end{Bmatrix} + 400 \times 10^3 \begin{Bmatrix} 0 \\ 0.0018 \end{Bmatrix}$$

$$\begin{Bmatrix} 301.47 \\ -416.67 \end{Bmatrix} = \begin{bmatrix} 320\,000 & 80\,000 \\ 80\,000 & 160\,000 \end{bmatrix} \begin{Bmatrix} \theta_b \\ \theta_c \end{Bmatrix} + \begin{Bmatrix} 0 \\ 720 \end{Bmatrix}$$

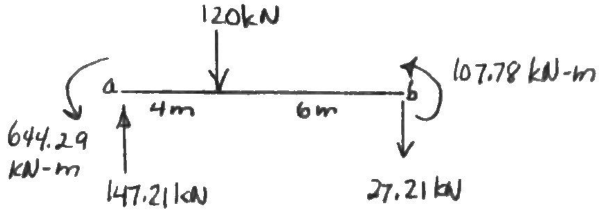
$$\begin{Bmatrix} 301.47 \\ -1136.67 \end{Bmatrix} = \begin{bmatrix} 320\,000 & 80\,000 \\ 80\,000 & 160\,000 \end{bmatrix} \begin{Bmatrix} \theta_b \\ \theta_c \end{Bmatrix} \Rightarrow \begin{Bmatrix} \theta_b \\ \theta_c \end{Bmatrix} = \begin{Bmatrix} 3.1064 \times 10^{-3}\,rad. \\ -8.6574 \times 10^{-3}\,rad \end{Bmatrix}$$

$\Delta_b = 0.03\,m \downarrow$

6) Compute member forces $\{f\}_m = [k]_m \{\Delta\}_m + \{f^F\}_m$

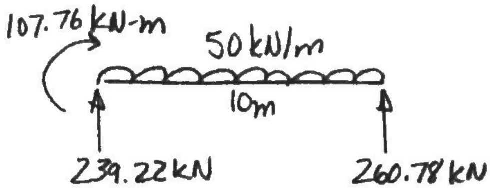
Member ①: $\{f\}_{a\text{-}b} = \begin{Bmatrix} M_a \\ M_b \\ V_a \\ V_b \end{Bmatrix} = \begin{bmatrix} 160000 & 80000 & 24000 & -24000 \\ 80000 & 160000 & 24000 & -24000 \\ 24000 & 24000 & 4800 & -4800 \\ -24000 & -24000 & -4800 & 4800 \end{bmatrix} \begin{Bmatrix} 0 \\ 3.1064\times10^{-3} \\ 0 \\ 0.03 \end{Bmatrix} + \begin{Bmatrix} -172.8 \\ 115.2 \\ -77.76 \\ -42.24 \end{Bmatrix}$

$= \begin{Bmatrix} -471.49 \\ -222.98 \\ -69.45 \\ 69.45 \end{Bmatrix} + \begin{Bmatrix} -172.8 \\ 115.2 \\ -77.76 \\ -42.24 \end{Bmatrix} = \begin{Bmatrix} -644.29\text{ kN-m} \circlearrowleft \\ -107.78\text{ kN-m} \circlearrowleft \\ -147.21\text{ kN} \uparrow \\ 27.21\text{ kN} \downarrow \end{Bmatrix}$

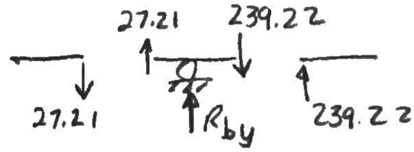


Member ②: $\{f\}_{b\text{-}c} = \begin{Bmatrix} M_b \\ M_c \\ V_b \\ V_c \end{Bmatrix} = \begin{bmatrix} 160000 & 80000 & 24000 & -24000 \\ 80000 & 160000 & 24000 & -24000 \\ 24000 & 24000 & 4800 & -4800 \\ -24000 & -24000 & -4800 & 4800 \end{bmatrix} \begin{Bmatrix} 3.1064\times10^{-3} \\ -8.6574\times10^{-3} \\ 0.03 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -416.67 \\ 416.67 \\ -250 \\ -250 \end{Bmatrix}$

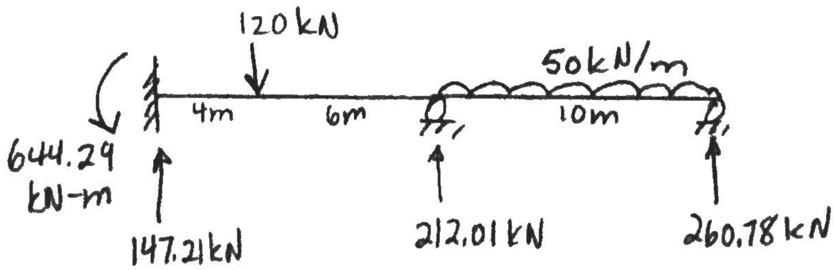
$= \begin{Bmatrix} 524.43 \\ -416.67 \\ 10.78 \\ -10.78 \end{Bmatrix} + \begin{Bmatrix} -416.67 \\ 416.67 \\ -250 \\ -250 \end{Bmatrix} = \begin{Bmatrix} 107.76\text{ kN-m} \circlearrowright \\ 0 \\ -239.22\text{ kN} \uparrow \\ -260.78\text{ kN} \uparrow \end{Bmatrix}$



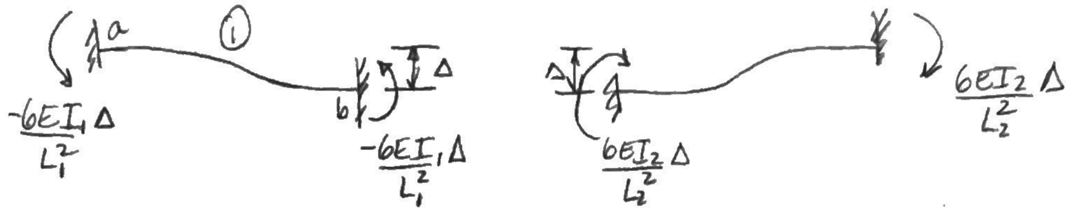
Reactions: joint b



$\Sigma F_y = 0: \; R_{by} - 239.22 + 27.21 = 0 \Rightarrow R_{by} = 212.01\text{ kN} \uparrow$



Problem 3 - Solution 2 - Treat settlement as a fixed-end moment & use 2x2 rotational stiffness matrix since dof's are θ_2 & θ_3



$$\frac{6EI_1}{L_1^2}\Delta = \frac{6(400\times10^3)}{(10)^2}(0.03) = 720\text{ kN-m} \qquad \frac{6EI_2}{L_2^2}\Delta = \frac{6(400\times10^3)}{(10)^2}(0.03) = 720\text{ kN-m}$$

$$\begin{Bmatrix} M_a^F \\ M_b^F \end{Bmatrix}_1 = \begin{Bmatrix} -172.8 \\ 115.2 \end{Bmatrix} + \begin{Bmatrix} -720 \\ -720 \end{Bmatrix} = \begin{Bmatrix} -892.8 \\ -604.8 \end{Bmatrix}; \quad \begin{Bmatrix} M_b^F \\ M_c^F \end{Bmatrix} = \begin{Bmatrix} -416.67 \\ 416.67 \end{Bmatrix} + \begin{Bmatrix} 720 \\ 720 \end{Bmatrix} = \begin{Bmatrix} 303.33 \\ 1136.67 \end{Bmatrix}$$

$$[k]_1 = \begin{array}{cc} \theta_a & \theta_b \end{array} \begin{bmatrix} 160000 & 80000 \\ 80000 & 160000 \end{bmatrix} \begin{array}{c} \theta_a \\ \theta_b \end{array} \qquad [k]_2 = \begin{array}{cc} \theta_b & \theta_c \end{array} \begin{bmatrix} 160000 & 80000 \\ 80000 & 160000 \end{bmatrix} \begin{array}{c} \theta_b \\ \theta_c \end{array}$$

$\{F\} = [K]\{\Delta\}$ (reduced eqs.)

$$\{F\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} (-604.8+303.33) \\ 1136.67 \end{Bmatrix} = \begin{Bmatrix} 301.47 \\ -1136.67 \end{Bmatrix}$$

$$\begin{Bmatrix} 301.47 \\ -1136.67 \end{Bmatrix} = \begin{bmatrix} 320000 & 80000 \\ 80000 & 160000 \end{bmatrix} \begin{Bmatrix} \theta_b \\ \theta_c \end{Bmatrix}$$

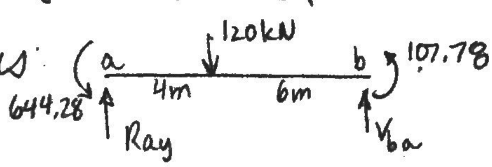
$$\Rightarrow \begin{Bmatrix} \theta_b \\ \theta_c \end{Bmatrix} = \begin{Bmatrix} 3.1064\times10^{-3}\text{ rad} \curvearrowright \\ -8.6574\times10^{-3}\text{ rad} \curvearrowleft \end{Bmatrix}$$

Member forces:

$$①: \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} 160000 & 80000 \\ 80000 & 160000 \end{bmatrix} \begin{Bmatrix} 0 \\ 3.1064\times10^{-3} \end{Bmatrix} + \begin{Bmatrix} -892.8 \\ -604.8 \end{Bmatrix} = \begin{Bmatrix} -644.28\text{ kN-m} \curvearrowleft \\ -107.78\text{ kN-m} \curvearrowleft \end{Bmatrix}$$

$$②: \begin{Bmatrix} M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} 160000 & 80000 \\ 80000 & 160000 \end{bmatrix} \begin{Bmatrix} 3.1064\times10^{-3} \\ -8.6574\times10^{-3} \end{Bmatrix} + \begin{Bmatrix} 303.33 \\ 1136.67 \end{Bmatrix} = \begin{Bmatrix} +107.76\text{ kN-m} \curvearrowright \\ 0 \end{Bmatrix}$$

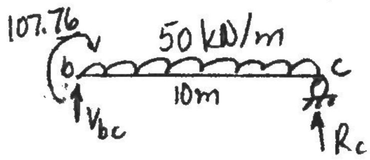
Reactions:



$\Sigma M_b = 0 \curvearrowright: 10R_{ay} - 644.28 - 107.78 - 120(6) = 0$

$R_{ay} = 147.21\text{ kN} \uparrow$

$\Sigma F_y = 0 \uparrow: V_{ba} = 27.21\text{ kN} \downarrow$



$\Sigma M_b = 0 \curvearrowright: -10R_{cy} + 107.76 + 50(10)(5) = 0$

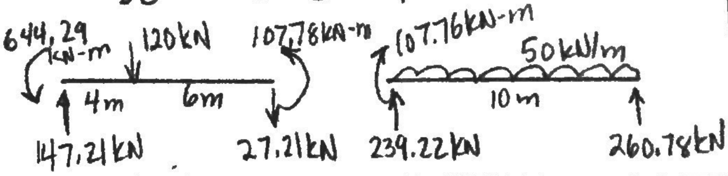
$\Rightarrow R_{cy} = 260.78\text{ kN} \uparrow$

$\Sigma F_y = 0: V_{bc} = 239.22\text{ kN} \uparrow$

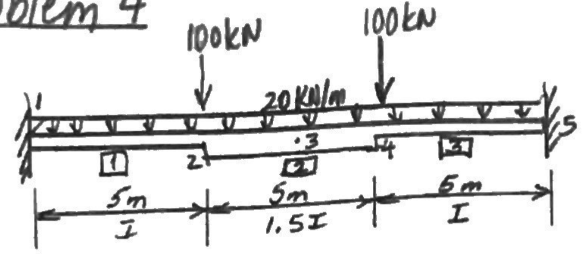
$\Sigma F_y = 0$ (whole beam)

$-120 - 50(10) + 147.21 + 260.78 + R_{by} = 0$

$\Rightarrow R_{by} = 212.01\text{ kN} \uparrow$

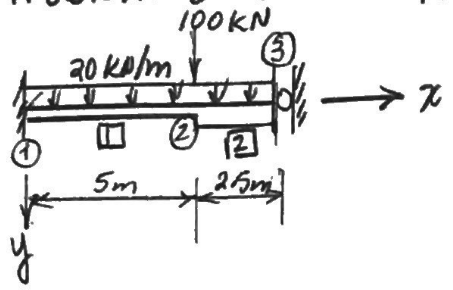


Problem 4



$E = 30\ GPa$ & $I = 4.8 \times 10^9\ mm^4$

1) Problem Details - Applying symmetry



DOF's : θ_2, Δ_{2y}, Δ_{3y}

$\theta_1 = 0$, $\Delta_{1y} = 0$
& $\theta_3 = 0$ because of symmetry

2) Calculate the fixed-end force vector for each member, $\{f^F\}_m$:

Member 1:

$$M_1^F = \frac{-wL^2}{12} = \frac{-20(5)^2}{12} = -41.67\ kN\text{-}m \ ; \quad M_2^F = +41.67\ kN\text{-}m$$

$$V_1^F = \frac{-wL}{2} = -\frac{20(5)}{2} = -50\ kN \uparrow \ ; \quad V_2^F = \frac{-wL}{2} = -50\ kN \uparrow$$

$$\{f^F\}_1 = \begin{Bmatrix} -41.67 \\ +41.67 \\ -50 \\ -50 \end{Bmatrix} \begin{matrix} M_1^F \\ M_2^F \\ V_1^F \\ V_2^F \end{matrix}$$

Member 2:

$$M_2^F = \frac{-wL^2}{12} = \frac{-20(2.5)^2}{12} = -10.41667\ kN\text{-}m \ ; \quad M_3^F = +10.41667\ kN\text{-}m$$

$$V_2^F = \frac{-wL}{2} = \frac{-20(2.5)}{2} = -25\ kN \uparrow \ ; \quad V_3^F = \frac{-wL}{2} = -25\ kN \uparrow$$

$$\{f^F\}_2 = \begin{Bmatrix} -10.41667 \\ 10.41667 \\ -25 \\ -25 \end{Bmatrix} \begin{matrix} M_2^F \\ M_3^F \\ V_2^F \\ V_3^F \end{matrix}$$

3) Calculate member stiffness matrix [k] for each member.
Since local & global axes coincide:

$$[k]_1 = [k']_1 = \frac{EI}{L^3}\begin{bmatrix} 4L^2 & 2L^2 & 6L & -6L \\ 2L^2 & 4L^2 & 6L & -6L \\ 6L & 6L & 12 & -12 \\ -6L & -6L & -12 & 12 \end{bmatrix}$$

$$= \frac{30\times10^6(4.8\times10^{-3}m^4)}{(5)^3}\begin{bmatrix} 100 & 50 & 30 & -30 \\ 50 & 100 & 30 & -30 \\ 30 & 30 & 12 & -12 \\ -30 & -30 & -12 & 12 \end{bmatrix}$$

1152

$$= \begin{array}{cc} \begin{matrix} \theta_1 & \theta_2 & \Delta_{1y} & \Delta_{2y} \end{matrix} & \\ \begin{bmatrix} 115200 & 57600 & 34560 & -34560 \\ 57600 & 115200 & 34560 & -34560 \\ 34560 & 34560 & 13824 & -13824 \\ -34560 & -34560 & -13824 & 13824 \end{bmatrix} & \begin{matrix} \theta_1 \\ \theta_2 \\ \Delta_{1y} \\ \Delta_{2y} \end{matrix} \end{array}$$

$$[k]_2 = [k']_2 = \frac{30\times10^6(1.5\times4.8\times10^{-3}m^4)}{(2.5)^3}\begin{bmatrix} 25 & 12.5 & 15 & -15 \\ 12.5 & 25 & 15 & -15 \\ 15 & 15 & 12 & -12 \\ -15 & -15 & -12 & 12 \end{bmatrix}$$

13824

$$= \begin{array}{cc} \begin{matrix} \theta_2 & \theta_2 & \Delta_{2y} & \Delta_{3y} \end{matrix} & \\ \begin{bmatrix} 345600 & 172800 & 207360 & -207360 \\ 172800 & 345600 & 207360 & -207360 \\ 207360 & 207360 & 165888 & -165888 \\ -207360 & -207360 & -165888 & 165888 \end{bmatrix} & \begin{matrix} \theta_2 \\ \theta_3 \\ \Delta_{2y} \\ \Delta_{3y} \end{matrix} \end{array}$$

4) Assemble global structure stiffness matrix [K] + nodal force vector {F}:

- Reduced stiffness matrix:

$$[K] = \begin{array}{cc} \begin{matrix} \theta_2 & \Delta_{2y} & \Delta_{3y} \end{matrix} & \\ \begin{bmatrix} (115200+345600) & (-34560+207360) & -207360 \\ (-34560+207360) & (13824+165888) & -165888 \\ -207360 & -165888 & 165888 \end{bmatrix} & \begin{matrix} \theta_2 \\ \Delta_{2y} \\ \Delta_{3y} \end{matrix} \end{array}$$

$$= \begin{bmatrix} 460800 & 172800 & -207360 \\ 172800 & 179712 & -165888 \\ -207360 & -165888 & 165888 \end{bmatrix}$$

$$\{F\} = \begin{Bmatrix} 0 \\ 100 \\ 0 \end{Bmatrix} - \begin{Bmatrix} (41.67 + (-10.41667)) \\ (-50 + (-25)) \\ -25 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 100 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 31.25 \\ -75 \\ -25 \end{Bmatrix} = \begin{Bmatrix} -31.25 \\ 175 \\ 25 \end{Bmatrix}$$

5) Impose b.c.'s & solve reduced set of eqs.

$$\begin{bmatrix} 460800 & 172800 & -207360 \\ 172800 & 179712 & -165888 \\ -207360 & -165888 & 165888 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \Delta_{2y} \\ \Delta_{3y} \end{Bmatrix} = \begin{Bmatrix} -31.25 \\ 175 \\ 25 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \theta_2 \\ \Delta_{2y} \\ \Delta_{3y} \end{Bmatrix} = \begin{Bmatrix} 0.004340278 \text{ rad} \circlearrowright \\ 0.0253183 \text{ m} \downarrow \\ 0.0308943 \text{ m} \downarrow \end{Bmatrix}$$

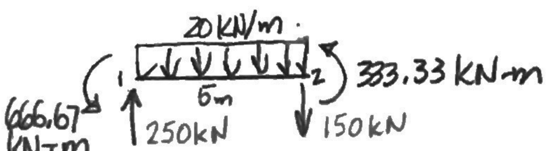
$\theta_4 = -0.004340278$ rad $\circlearrowleft$ & $\Delta_{4y} = 0.0253183$ m $\downarrow$ (from reflective symmetry)

6) Determine forces in each member and reactions:

Member 1: $\{f\}_{1\text{-}2} = [k]_1\{\Delta\} + \{f^F\}_1$

$$\begin{Bmatrix} M_1 \\ M_2 \\ V_1 \\ V_2 \end{Bmatrix} = \begin{bmatrix} 115200 & 57600 & 34560 & -34560 \\ 57600 & 115200 & 34560 & -34560 \\ 34560 & 34560 & 13824 & -13824 \\ -34560 & -34560 & -13824 & 13824 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.004340278 \\ 0 \\ 0.0253183 \end{Bmatrix} + \begin{Bmatrix} -41.67 \\ +41.67 \\ -50 \\ -50 \end{Bmatrix}$$

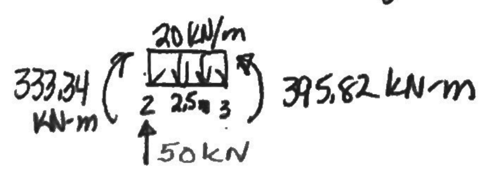
$$= \begin{Bmatrix} -625 \\ -375 \\ -200 \\ 200 \end{Bmatrix} + \begin{Bmatrix} -41.67 \\ 41.67 \\ -50 \\ -50 \end{Bmatrix} = \begin{Bmatrix} -666.67 \text{ kN-m} \\ -333.33 \text{ kN-m} \\ -250 \text{ kN} \uparrow \\ 150 \text{ kN} \downarrow \end{Bmatrix}$$



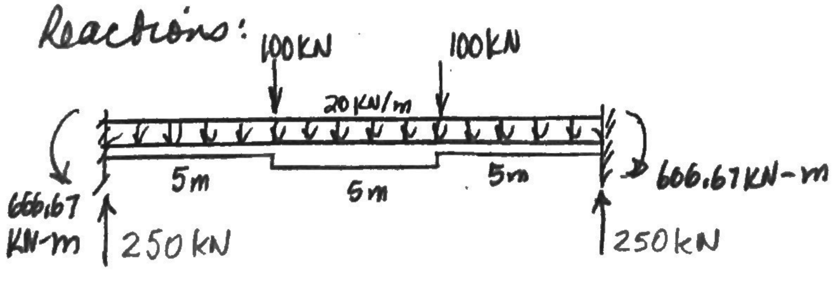
Member 2: $\{f\}_{2\text{-}3} = [k]_2\{\Delta\} + \{f^F\}_2$

$$\begin{Bmatrix} M_2 \\ M_3 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{bmatrix} 345600 & 172800 & 207360 & -207360 \\ 172800 & 345600 & 207360 & -207360 \\ 207360 & 207360 & 165888 & -165888 \\ -207360 & -207360 & -165888 & 165888 \end{bmatrix} \begin{Bmatrix} 0.004340278 \\ 0 \\ 0.0253183 \\ 0.0308943 \end{Bmatrix} + \begin{Bmatrix} -10.4167 \\ 10.4167 \\ -25 \\ -25 \end{Bmatrix}$$

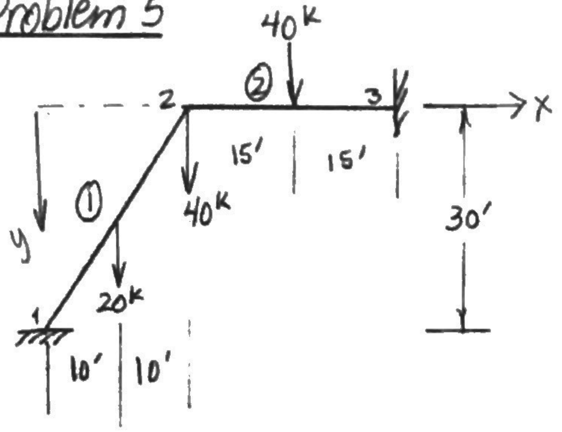
$$= \begin{Bmatrix} 343.7607 \\ -406.24 \\ -25 \\ 25 \end{Bmatrix} + \begin{Bmatrix} -10.4167 \\ 10.4167 \\ -25 \\ -25 \end{Bmatrix} = \begin{Bmatrix} 333.34 \text{ kN-m} \\ -395.82 \text{ kN-m} \\ -50 \text{ kN} \uparrow \\ 0 \end{Bmatrix}$$



Reactions:



Problem 5

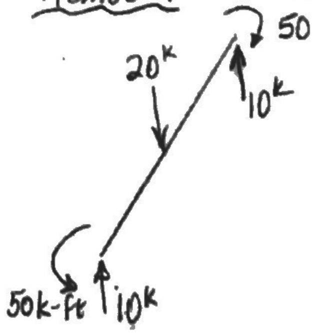


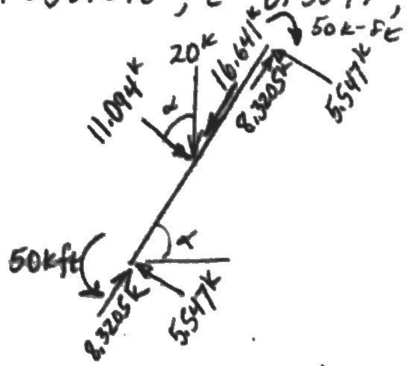
$E = 30000$ ksi , $A = 10 in^2$, $I = 200 in^4$

1) Problem Details
- Number joints & members
- Dof's : $\Delta_{2x}, \Delta_{2y}, \theta_2$

2) Calculate Fixed-End Forces

Member 1 $L = 36.056$ ft , $\alpha = 56.310°$, $C = 0.5547$, $s = 0.83205$





$FEM_{12} = \dfrac{11.094(36.056)}{8}$

$= 50$ k-ft

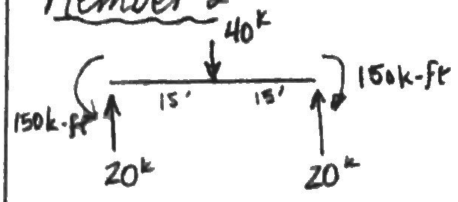
in global coord.

$$\begin{array}{c} \begin{Bmatrix} 0 \\ -10 \\ -50 \\ 0 \\ -10 \\ +50 \end{Bmatrix} \end{array} \begin{array}{l} \Delta_{1x} \\ \Delta_{1y} \\ \theta_1 \quad \text{k-ft} \\ \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \quad \text{k-ft} \end{array}$$

in local coord.

$$\begin{Bmatrix} -50 \\ +50 \\ -5.547 \\ -5.547 \\ 8.3205 \\ 8.3205 \end{Bmatrix} \begin{array}{l} \theta_1 \\ \theta_2 \\ \Delta_1 \\ \Delta_2 \\ \delta_1 \\ \delta_2 \end{array}$$

Member 2



in global coord.

$$\begin{Bmatrix} 0 \\ -20 \\ -150 \\ 0 \\ -20 \\ +150 \end{Bmatrix} \begin{array}{l} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \quad \text{k-ft} \\ \Delta_{3x} \\ \Delta_{3y} \\ \theta_3 \quad \text{k-ft} \end{array}$$

in local coord

$$\begin{Bmatrix} -150 \\ +150 \\ -20 \\ -20 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{l} \theta_2 \\ \theta_3 \\ \Delta_{2y} \\ \Delta_{3y} \\ \delta_2 \\ \delta_3 \end{array}$$

3) Calculate member stiffness matrix [k] in global coord. (Eq. 18.81) - for simplicity write only Coeff. for dof's Δ_{2x}, Δ_{2y} & θ_2

Member 1: $\alpha = 56.310^\circ$, $c = 0.554$, $s = 0.83205$, $N = \frac{A}{I} = \frac{10}{200} = 0.05 in^{-2}$;

$$P = \frac{12}{L^2} = \frac{12}{(36.056 \times 12)^2} = 6.410 \times 10^{-5} in^{-2};\ Q = \frac{6}{L} = \frac{6}{(36.056 \times 12)} = 0.0138675 in^{-1}$$

$$\frac{EI}{L} = \frac{30000(200)}{(36.056 \times 12)} = 13867.5 \text{ k-in}$$

$$[k]_1 = \begin{array}{cc} \begin{bmatrix} 213.42 & -319.21 & -160.00 \\ -319.21 & 480.30 & -106.54 \\ -160.00 & -106.54 & 55470 \end{bmatrix} & \begin{array}{c} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{array} \end{array}$$

Member 2: $\alpha = 0$, $c = 1$, $s = 0$, $N = 0.05 in^2$, $P = \frac{12}{(30 \times 12)^2} = 9.25926 \times 10^{-5} in^{-2}$,

$$Q = \frac{6}{L} = \frac{6}{(30 \times 12)} = 0.016667 in^{-1};\ \frac{EI}{L} = \frac{30000(200)}{(30 \times 12)} = 16666.67 \text{ k-in}$$

$$[k]_2 = \begin{array}{cc} \begin{bmatrix} 833.33 & 0 & 0 \\ 0 & 1.54321 & 277.78 \\ 0 & 277.78 & 66666.67 \end{bmatrix} & \begin{array}{c} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{array} \end{array}$$

4) Assemble global structure stiffness matrix [K] & nodal force vector {F}

$$\{F\} = \underbrace{\begin{Bmatrix} 0 \\ 40^k \\ 0 \end{Bmatrix}}_{\text{Ext. nodal loads}} - \underbrace{\begin{Bmatrix} 0 \\ (-10)+(-20) \text{ k} \\ 50+(-150) \text{ k-ft} \end{Bmatrix}}_{\text{Fixed-end forces}} = \begin{Bmatrix} 0 \\ 40 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -30 \\ -100 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 70^k \\ 100 \text{ k-ft} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 70^k \\ 1200 \text{ k-in} \end{Bmatrix}$$

$$[K] = \begin{bmatrix} (213.42+833.33) & -319.21 & -160.00 \\ -319.21 & (480.30+1.54321) & (-106.54+277.78) \\ -160.00 & (-106.54+277.78) & (55470+66667) \end{bmatrix}$$

$$= \begin{bmatrix} 1046.75 & -319.21 & -160.00 \\ -319.21 & 481.84 & 171.24 \\ -160.00 & 171.24 & 122137 \end{bmatrix}$$

5) Impose b.c.'s & solve eqs. to obtain unknown displ.

$$\begin{bmatrix} 1046.75 & -319.21 & -160.00 \\ -319.21 & 481.84 & 171.24 \\ -160.00 & 171.24 & 122137 \end{bmatrix} \begin{Bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 70^k \\ 1200\text{k-in} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.0561 \text{ in} \rightarrow \\ 0.1790 \text{ in.} \downarrow \\ 0.0096475 \text{ rad.} \end{Bmatrix}$$

6) Compute Member forces, $\{f\}_m = [\underbrace{k'}_{\text{local coord.}}]_m \{\underbrace{\delta}_{\text{local coord.}}\}_m + \{f^F\}_m$

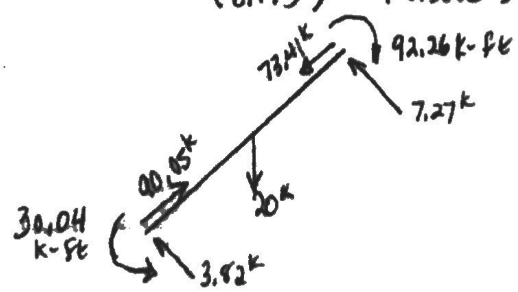
<u>Member 1</u>: $\{\delta\}_1 = [T]\{\Delta\}_1$

$$\{\delta\}_1 = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \Delta_1 \\ \Delta_2 \\ \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S & C & 0 \\ C & -S & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & -S & 0 \end{bmatrix} \begin{Bmatrix} \Delta_{1x} \\ \Delta_{1y} \\ \theta_1 \\ \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0.83205 & 0.5540 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.83205 & 0.5540 & 0 \\ 0.5540 & -0.83205 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5540 & -0.83205 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0561 \\ 0.1790 \\ 0.0096475 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.0096475 \\ 0 \\ 0.14580 \text{ in} \\ 0 \\ -0.11787 \text{ in} \end{Bmatrix}$$

$$[k']_1 = \begin{bmatrix} 4EI/L & 2EI/L & 6EI/L^2 & -6EI/L^2 & 0 & 0 \\ & 4EI/L & 6EI/L^2 & -6EI/L^2 & 0 & 0 \\ & & 12EI/L^3 & -12EI/L^3 & 0 & 0 \\ & \text{Symm.} & & 12EI/L^3 & 0 & 0 \\ & & & & AE/L & -AE/L \\ & & & & & AE/L \end{bmatrix} = \begin{bmatrix} 55470 & 27735 & 192.31 & -192.31 & 0 & 0 \\ & 55470 & 192.31 & -192.31 & 0 & 0 \\ & & 0.8889 & -0.8889 & 0 & 0 \\ & & & 0.8889 & 0 & 0 \\ & & & & 693.38 & -693.38 \\ & & & & & 693.38 \end{bmatrix}$$

$$\{f\}_1 = [k']_1\{\delta\}_1 + \{f^F\}_1 = \begin{Bmatrix} 239.53 \\ 507.10 \\ 1.726 \\ -1.726 \\ 81.73 \\ -81.73 \end{Bmatrix} + \begin{Bmatrix} -600 \\ +600 \\ -5.547 \\ -5.547 \\ 8.3205 \\ 8.3205 \end{Bmatrix} = \begin{Bmatrix} -360.47 \text{ k-in} = -30.04 \text{ k-ft} \\ 1107.10 \text{ k-in} = 92.26 \text{ k-ft} \\ -3.82 \text{ k} \uparrow \\ -7.27 \text{ k} \uparrow \\ 90.05 \text{ k} \rightarrow \\ -73.41 \text{ k} \leftarrow \end{Bmatrix}$$



Member 2:

$$\{f\}_2 = [k']_2\{\delta\}_2 + \{f^F\}_2$$

$$\{\delta\}_2 = \begin{Bmatrix} \theta_2 \\ \theta_3 \\ \Delta_2 \\ \Delta_3 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} 0.0096475 \\ 0 \\ 0.1790 \\ 0 \\ 0.0561 \\ 0 \end{Bmatrix}$$

$$\{f\}_2 = \begin{bmatrix} 66667 & 33333 & 277.78 & -277.78 & 0 & 0 \\ & 66667 & 277.78 & -277.78 & 0 & 0 \\ & & 1.5432 & -1.5432 & 0 & 0 \\ & \text{Symm.} & & 1.5432 & 0 & 0 \\ & & & & 833.33 & -833.33 \\ & & & & & 833.33 \end{bmatrix} \begin{Bmatrix} 0.0096475 \\ 0 \\ 0.1790 \\ 0 \\ 0.0561 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -1800 \text{ k-in} \\ 1800 \text{ k-in} \\ -20 \text{ k} \\ -20 \text{ k} \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 692.89 \\ 371.31 \\ 2.956 \\ -2.956 \\ 46.75 \\ -46.75 \end{Bmatrix} + \begin{Bmatrix} -1800 \\ 1800 \\ -20 \\ -20 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1107.1 \text{ k-in} = -92.26 \text{k-ft} \circlearrowright \\ 2171.3 \text{ k-in} = 180.94 \text{k-ft} \circlearrowright \\ -17.04 \text{ k} \uparrow \\ -22.96 \text{ k} \uparrow \\ 46.75 \text{ k} \rightarrow \\ -46.75 \text{ k} \leftarrow \end{Bmatrix}$$