

Name: \_\_\_\_\_

Extra Credit Value: 25 Points

**Eigenvalue Worksheet**  
Due: November 17, 2018

Consider the  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (1)$$

We know some things.

1.  $\det(A) = ad - bc = \lambda_1 \lambda_2$  and if  $\det(A) \neq 0$  then  $A$  is invertible.

**Question 1:** What does this suggest about the relationship between a matrix's eigenvalues and whether  $A^{-1}$  exists?

2.  $\text{trace}(A) = a + d = \lambda_1 + \lambda_2$ .

**Note:** This does NOT mean that  $a = \lambda_1$  or that  $d = \lambda_2$ ; but if we know  $a, d$ , and  $\lambda_1$  we can find  $\lambda_2$ .

3. The characteristic polynomial in a variety of forms

$$\begin{aligned} p_A(t) &= \det(A - tI) = \begin{vmatrix} a-t & b \\ c & d-t \end{vmatrix} = (a-t)(d-t) - bc \\ &= t^2 - at - dt + ad - bc = t^2 - \underbrace{(a+d)}_{\text{trace}(A)}t + \underbrace{ad - bc}_{\det(A)} \\ \Rightarrow p_A(t) &= t^2 - (\lambda_1 + \lambda_2)t + \lambda_1 \lambda_2 \end{aligned} \quad (2)$$

4. Then, solving for  $t$ , we get

$$\begin{aligned} t &= \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{(a+d) \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4bc}}{2} \\ &= \frac{(a+d) \pm \sqrt{a^2 - 2ad + d^2 + 4bc}}{2} \end{aligned}$$

Then, if  $a = d$  and  $b = c$ , we have

$$t = \frac{2a \pm \sqrt{a^2 - 2a^2 + a^2 + 4b^2}}{2} = \frac{2a \pm \sqrt{4b^2}}{2} = a \pm b.$$

And if  $a = d$  and  $b = -c \Rightarrow c = -b$ , we have

$$t = \frac{2a \pm \sqrt{a^2 - a^2 + a^2 - 4b^2}}{2} = \frac{2a \pm \sqrt{-4b^2}}{2} = a \pm b\sqrt{-1} = a \pm bi$$

- (a) **Question 2:** Find  $t$  when  $a = b = c = d$ .

(b) **Question 3:** Find  $t$  when  $a = c$  and  $b = d$ .

5. To find the eigenvectors of  $A$  we note that finding eigenvectors is just a nullspace calculation. That is, we need to find a non-trivial  $\mathbf{x}$  such that  $(A - \lambda_i I)\mathbf{x} = \mathbf{0}$ . Or,

$$(A - \lambda_i I) = \begin{bmatrix} a - \lambda_i & b \\ c & d - \lambda_i \end{bmatrix}, \quad i = 1, 2.$$

We can solve in two ways

(a) *RREF*.

(b) Or,

$$\begin{aligned} (A - \lambda_i I)\mathbf{x} = \mathbf{0} &\Rightarrow \begin{bmatrix} a - \lambda_i & b \\ c & d - \lambda_i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow (a - \lambda_i)x_1 + bx_2 = 0 \\ &\Rightarrow cx_1 + (d - \lambda_i)x_2 = 0 \end{aligned}$$

6. **Example 1:** Let  $A \in \mathbb{R}^{2 \times 2}$  be

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

• Then, solving for the eigenvalues, we have

$$\det(A - tI) = \begin{vmatrix} -t & 1 \\ -2 & -3 - t \end{vmatrix} = -t(-3 - t) - 1(-2) = t^2 + 3t + 2 = (t + 1)(t + 2)$$

Thus,  $\lambda_1 = -1$ , and  $\lambda_2 = -2$

And  $am(-1) = 1$ , and  $am(-2) = 1$

• Now onto the eigenvectors. First way

$$\begin{aligned} A - (-1)I &= \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \xrightarrow{2R_1 + R_2 \rightarrow R_2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ &\Rightarrow \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &\Rightarrow E(-1) = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, \text{ and } gm(-1) = 1 \end{aligned}$$

• The second way

$$\begin{aligned} (A - (-2)I)\mathbf{x} = \mathbf{0} &\Rightarrow \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow 2x_1 + x_2 = 0 \text{ (Since one row is a scaling of the other we can use either, but only one is needed)} \\ &\Rightarrow 2x_1 = -x_2 \Rightarrow x_1 = -\frac{1}{2}x_2 \\ &\Rightarrow \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow E(-2) = \text{Span} \left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\} \Rightarrow gm(-2) = 1 \end{aligned}$$

Question 4: For the matrix

$$A = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$$

- Find the eigenvalues, using what you've derived in (a).
- Find the eigenvectors using either method.
- What are the algebraic and geometric multiplicities of each eigenvalue?
- Is  $A$  invertible? Use the eigenvalues for your reason why.

7. Now, some fun facts

(a) For any matrix  $A \in \mathbb{R}^{n \times n}$  we have

$$AP = PD. \quad (3)$$

where  $D$  is a diagonal matrix with the eigenvalues of  $A$  on its main diagonal and  $P$  is a matrix made up of the eigenvectors of  $A$ .

**Question 5:** Is  $P$  unique for  $A$ ? Why or why not?

(b) If  $A$  has  $n$  linearly independent eigenvectors then  $A$  is diagonalizable. This is because if  $A$  fulfills the prior sentence, the matrix made of its eigenvectors  $P$  will be row equivalent to the identity matrix  $I_n$ .

**Question 6:** Let  $A \in \mathbb{R}^{n \times n}$  and let  $A$  have one distinct eigenvalue  $\lambda_1$  repeated  $n$  times. Thus the algebraic multiplicity of  $\lambda_1$  is  $am(\lambda_1) = n$ . If the geometric multiplicity of  $\lambda_1$  is strictly less than its algebraic multiplicity ( $am(\lambda_1) > gm(\lambda_1)$ ) is  $A$  diagonalizable? Why or why not?

(c) An  $n \times n$  matrix with  $n$  distinct eigenvalues is diagonalizable.

**Question 7:** Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Obviously, the eigenvalues of  $A$  are  $\lambda_2 = 2$  and  $\lambda_1 = \lambda_3 = 1$ . Is  $A$  diagonalizable? Why or why not?

**Hint:** get the eigenvectors of  $A$ .

(d) If  $A$  is diagonalizable, we can easily calculate  $A^k$  for any natural number  $k$ . We depend on the fact that since  $P$  is invertible and thus  $P^{-1}$  exists, we have  $PP^{-1} = P^{-1}P = I$ . Thus, we can find  $A^k$  by

$$\begin{aligned} A^k &= \underbrace{(PDP^{-1})(PDP^{-1})\dots(PDP^{-1})}_{k \text{ times}} \\ &= \underbrace{PD}_{=I} \underbrace{P^{-1}P}_{=I} \dots \underbrace{P^{-1}P}_{=I} \underbrace{PD}_{=I} P^{-1} \\ &= P \underbrace{DD\dots D}_{k \text{ times}} P^{-1} \\ &= PD^k P^{-1}. \end{aligned}$$

**Example 2:** Find  $A^5$  for the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

First, we have to find the eigenvalues. Which is easy given the form  $\lambda_1 = 4$  and  $\lambda_2 = 2$  giving us

$$D = \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \lambda_2 \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Next, the eigenvectors.

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{-R_1 + R_2 \rightarrow R_2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{-R_1 \rightarrow R_2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \xrightarrow{-R_1 + R_2 \rightarrow R_2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{Giving us } P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \text{ and } P^{-1} = -\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Now all we need do is compute  $A^5$ .

$$\begin{aligned} A^5 &= PD^5P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^5 & 0 \\ 0 & 2^5 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1024 & 0 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -1024 & 32 \\ 1024 & 32 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 528 & -496 \\ -496 & 528 \end{bmatrix} \end{aligned}$$

Question 8: Find  $A^6$  for the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

**Question 9:** Let  $A \in \mathbb{R}^{n \times n}$  and let the eigenvalues of  $A$  be between 0 and 1 (not including zero or one. In other words  $\lambda_i \in (0, 1) \forall i$ .  $(0, 1)$  is called an "open interval"). What will happen to  $A^k$  as  $k$  grows increasingly large?

**Question 10:** Find  $B^2, B^4, B^8, B^{16}$ , and  $B^{32}$  for

$$B = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

Use a computer if you need it, but use the method above.

8. An idempotent matrix  $A \in \mathbb{R}^{n \times n}$  is one in which  $A^2 = A$ . If  $A$  is an idempotent matrix, then  $A$  has eigenvalues of either 0 or 1.

Question 11: Let  $A, B$ , and  $C$  be

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Find the eigenvalues of  $A, B$ , and  $C$ .

- Which ones are idempotent? Why?

- Which ones are invertible?

A has

9. A matrix  $A \in \mathbb{R}^{n \times n}$  is nilpotent if  $AA = 0$ . If  $A$  is a nilpotent matrix, then  $A$  has only values of 0 as its eigenvalues.

Question 12: For the matrix below,

$$N = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

- Show that  $N$  is a nilpotent matrix.

- Find the eigenvalues of  $N$

- Find the eigenvectors of  $N$ .