CHAPTER 12

Strategies in Mathematics

Mathematics is a core academic competency. Mastering basic mathematic skills (e.g., knowledge of basic facts, problem solving) is critical for successful functioning in society. We are constantly confronted with problems that involve mathematics skills. These might range from "Do I have enough money to pay for these three items I want to purchase?" to "I have to go 30 miles to a meeting that takes place in 30 minutes—how fast do I have to drive to get there on time?" Mastering the skills and strategies necessary to solve these problems is critical for functioning effectively in society and in the workplace (Chesloff & Maier, 2012; Patton, Cronin, Bassett, & Koppel, 1997). Over the last 10 years, math achievement has steadily increased (National Center for Education Statistics, 2011); however, our level of mathematics achievement is still a matter of national concern, as research suggests that American children have serious deficits in many areas (Jitendra & Xin, 1997; Xin & Jitendra, 1999).

For the purposes of this chapter, we limit our discussion to three major areas of mathematics: basic facts, computation procedures, and word-problem solving. Knowledge of basic math facts is a foundational for more advanced operations. Fluency with mathematics requires students to learn and store basic facts (e.g., $3 + 5 = 8$; $9 \times 7 = 63$) in long-term memory, and then to rapidly and accurately recall and apply them. Computational procedures are also fundamental. Students must master procedures involved in computations such as long division, place value, and regrouping. They must also remember and apply these procedures correctly while simultaneously recalling and utilizing basic facts. Finally, students are required to develop and utilize word-problem-solving skills. Here students are confronted with word problems that require them to identify important information, conceptualize how to solve the problem, define appropriate equations, and solve equations. Again, knowledge of basic facts and computational procedures must also be accessed and utilized.

PROBLEMS FOR STUDENTS WITH LEARNING DISABILITIES

Math disabilities (MD; also termed *dyscalculia* or *developmental dyscalculia*) are a serious educational problem. Studies suggest that 5–10% of all school-age children have some type of serious deficit in mathematics (Murphy, Mazzocco, Hanich, & Early, 2007). MD are independent of intelligence in that students with very high IQs may also have math disabilities (Butterworth & Relgosa, 2007). MD may occur in isolation (i.e., the only difficulties are in math) or in combination with reading disabilities. It is thought that the problems of students with both MD and reading disabilities are due to underlying language deficits that may or may not be accompanied by numerical deficits (Jordan, 2007).

Difficulties experienced by children with MD span all three areas (basic facts, computation procedures, and problem solving). Children with MD are extremely heterogeneous; however, some problems commonly occur (Geary, 2003). These include:

- Poor understanding of concepts underlying mathematical procedures
- Problems with counting and using counting to solve problems
- Use of immature or inappropriate strategies
- Difficulty with fact retrieval
- Difficulty coordinating and monitoring steps in computation
- Difficulty in word-problem solving
- Working memory deficits

On the whole, children with LD are only slightly behind typically achieving peers in terms of development of number concepts (Geary, Hamson, & Hoard, 2000). But some commonly occurring problems, such as difficulty with counting concepts, have been demonstrated (Geary, Bow-Thomas, & Yao, 1992; Geary, Hoard, & Hamson, 1999). For example, children may believe that nonsequential counting (i.e., skipping some items and counting them later) will result in an incorrect count (Geary, 2003). Another example would be failure to grasp the commutative principle. Students who did not understand this principle would not understand that 8×5 is the same problem as 5×8. Poor understanding of number concepts can inhibit the use of advanced strategies and may affect ability to detect procedural errors (Geary, 2003). Counting problems are a serious concern because counting serves as an important preskill in the mastery of basic addition facts. Difficulty counting can inhibit the development of basic addition facts. Children with LD may be developmentally delayed in the use of counting to solve arithmetic problems. They may also fail to perceive the link between counting and problem solving. Many see counting as simply a rote activity (Geary, 2003).

Use of appropriate strategies is also a concern. Many children either fail to develop or fail to utilize more advanced strategies. For example, many children with LD don't gradually switch from counting to direct retrieval of math facts (Geary, Widaman, Little, & Corimer, 1987). Thus, these children use an inefficient means of retrieval. Memory problems are a major factor in poor mathematic performance. Children with LD commonly produce more errors in math fact retrieval than their peers. Research suggests that children with LD may have problems in storing basic facts in long-term memory and accessing them readily. Errors in retrieval may be due to problems inhibiting irrelevant information and/or associations (Geary et al., 2000). For example, with a problem such as 6 + 2, many will answer 7, the next number after 6. For complex problems (i.e., those that involve a number of operations), children with LD frequently make procedural errors. They may omit steps (e.g., fail to regroup) or have difficulty performing steps in the correct sequence. Procedural errors in computation appear to be due to problems in monitoring and coordinating the sequence of problem-solving steps (Russell & Ginsberg, 1984). Problems with working memory are common and are thought to be related to procedural difficulties (Geary, Hoard, Nugent, & Byrd-Craven, 2007).

Solving word problems is often extremely difficult for children with LD. The difficulties with word-problem solving go well beyond problems with decoding. Parmar, Cawley, and Frazita (1996) found that students with LD had difficulties representing problems, identifying salient information, and choosing the appropriate arithmetic operations. Word problems may contain irrelevant information, and solving them may require multiple operations or steps that can also cause serious problems (Fuchs & Fuchs, 2003). Thus, word problems place a number of demands on children's cognitive processing. These problems are exacerbated by the fact that mathematics instruction for children with LD often is focused on memorization of facts and computation skills at the expense of instruction on application of mathematics in problem-solving situations (Bottge, 1999; Parmar et al., 1994).

PREREQUISITE SKILLS

A list of all the possible prerequisite skills for mathematics would be beyond the scope of this chapter. There are, however, three major areas of preskills that are extremely important for successful development of mathematical skills: number sense, basic mathematics principles, and basic facts rules in mathematics. These prerequisites are critical to the development of skill with basic facts, computation procedures, and problem solving. In this section we discuss each preskill.

Number Sense

Number sense is a concept that refers to "a child's fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and look at the world and make comparisons" (Gersten & Chard, 1999, p. 20). Children with a good number sense have a "feel" for math. They can see patterns in numbers and have mastered concepts such as *greater than, less than,* and *equal to.* Van de Walle (1998) lists five components of number sense:

1. Well-understood number meanings (e.g., *3* corresponds to a certain quantity)
2. Awareness of multiple relationships among numbers (e.g., a set of *6* objects can be made with *2* sets of *3* or a set of *4* plus a set of *2*)
3. Recognition of the relative magnitude of numbers (e.g., *5* is larger than *2*)
4. Knowledge of the effects of operations on numbers (e.g., adding makes the number bigger)
5. Knowledge that numbers measure things in the real world

One of the key components of a good number sense is a well-developed counting ability. Counting involves the ability to produce number words and the knowledge that the number words have a one-to-one relationship with the set being counted. Counting is also a critical preskill for the development of knowledge of basic addition facts. Mastery of basic addition facts initially depends on counting-based strategies (Garnett, 1992). Children go through four stages in the mastery of basic addition facts:

1. *Count all*: Given 3 + 2, the student counts 1, 2, 3, 4, 5 to get the answer (i.e., counts all the numbers).
2. *Count on*: Given 2 + 5, the student begins at one addend and counts 2, 3, 4, 5, 6, 7.
3. *Count on from larger addend or shortcut sum*: Given 2 + 5, the student begins at the larger addend and counts 5, 6, 7.
4. *Memory*: Given 2 + 5, the student just "knows" the answer.

Obviously children with impaired counting skills will have difficulty progressing to the memory level where knowledge is automatic. Counting skills are also used in strategies for teaching basic facts, as we discuss later. Garnett (1992) recommended frequent practice in counting. This includes counting by 1's, 2's, and 5's. After students can count forward fluently, teachers should also attend to counting backward to provide the groundwork for subtraction skills.

Basic Mathematics Principles

A number of critical principles are considered foundational for mathematical understanding and apply to many different levels of mathematics. Harniss, Carnine, Silbert, and Dixon (2002) noted the following set of critical principles:

- *Place value*: The position of a number provides information about the value of the number.
- *Expanded notation*: Numbers can be reduced to their underlying units (e.g., the number 437 is equal to four 100's plus three 10's, plus seven 1's).
- *Commutative property*: The order of numbers in an equation does not affect the result (e.g., 8 + 7 = 7 + 8). This is true for addition and multiplication, but not subtraction and division.
- *Associative property*: The grouping of numbers in an equation can be changed without changing the result; for example, (8 + 7) + 4 = 8 + (7 + 4). Again, only addition and multiplication are associative.
- *Distributive property*: Numbers in an equation can be redistributed; for example, 7 × (8 + 4) = (7 × 8) + (7 × 4).
- *Equivalence*: The quantity on one side of the = sign is equal to the quantity on the other.

These principles, which Harniss and his colleagues (2002) term "big ideas," cut across all of basic mathematics and are critical for understanding mathematics and solving equations. For example, the commutative principle means that 5 × 4 is the same as 4 × 5. This, in turn, means that rather than having to memorize 100 basic addition facts, it's only necessary to learn 50, which decreases the memory burden significantly. Many children with LD do not understand the commutative principle and would think that the two problems were actually different.

Basic Facts Rules

For each of the four operations (addition, subtraction, multiplication, and division) there is an underlying set of rules. These rules are helpful in mastering the basic facts because they help to "chunk" facts into groups that are related. This chunking, in turn, can help reduce the demands on working memory and enable long-term storage in memory. These chunks can also be utilized as the basis for strategies, as we show later. Figure 12.1 shows the basic facts rules for addition, subtraction, multiplication, and division.

INSTRUCTION IN MATHEMATICS

In the sections that follow, we present examples of strategies for basic facts, computation procedures, and word-problem solving. We selected basic facts because they are the foundation for all mathematics skills; students must master basic facts if they are to progress in mathematics. Computation involves application of basic facts to solve more advanced problems, which requires students to learn and follow an effective procedure. Many children with LD have difficulty following a procedure. Word-problem solving was selected because it is a problem area for many children, and it is a critical area because it can help children develop problem-solving skills for real-world problems.

BASIC MATH FACTS STRATEGIES

Addition

Addition facts are the first in a series of fact families that must be learned. Basic facts are often taught by rote repetition—an extremely inefficient method for several reasons. First, it requires learning 100 facts by rote and thus ignores the commutative principle. Second, and more importantly, it does not provide any framework to help students organize information to be learned. As we noted earlier, facts that are organized and related are much easier to learn than unrelated facts. Thornton and Toohey (1985) developed a chunking strategy to simplify instruction in basic addition facts. This is an unstructured strategy that takes advantage of basic facts rules and simple mnemonics to group math facts by the strategy needed for recall. Chunking the addition facts into groups based on the strategy used to recall the addition fact makes is easier for students to recall the facts. These are the groups:

- *Count-ons.* These are the facts that involve adding 1 or 2 (e.g., 5 + 1 or 6 + 2). Students are prompted to start big, count on, and to "feel the count."
- *Zeros.* These are the facts that involve adding 0 (e.g., 3 + 0 or 8 + 0). For this set, students are taught that plus 0 stays the same.

Addition	Subtraction	Multiplication	Division
The Order Rule (or commutative principle)	*The 0 Rule*: any number minus 0 is the number.	*The 0 Rule*: any number times 0 equals 0.	*The 0 Rule*: 0 divided by any number is 0.
The 0 Rule: any number plus 0 is the number.	*The 1's Rule*: any number minus 1 is 1 less than the number (count backwards by 1).	*The 1's Rule*: any number times 1 equals the number.	*The 1's Rule*: any number divided by 1 is the number.
The 1's Rule: any number plus 1 is 1 more than the number.	*The Same Number Rule*: any number minus itself is 0.	*The 2's Rule*: any number times 2 is double the number. Thus 8 × 2 is equal to 8 + 8.	*The 2's Rule*: any number divided by 2 is half the number.
The 9's Rule: any single-digit number greater than 0 added to 9 results in the addend number minus 1 plus 10.	*The Addition/Subtraction Relationship Rule*: in a subtraction problem, the answer added to the number being subtracted equals the top number. Thus to solve 12 – 9, teach the student "What number plus 9 equals 12?"	*The 5's Rule*: any number times 5 is equivalent to counting by 5's the number of times indicated by the multiplier. Thus 5 × 4 is counting by 5's four times.	*The 9's Rule*: when dividing by 9 the answer is 1 more than the number in the 10's column. For example, 54 ÷ 9 = 6 (6 is 1 more than the number in the 10's column).
The 10's Rule: any single-digit number added to 10 results in the 0 being changed to the number being added.		*The 9's Rule*: when multiplying by 9, the answers can be found by taking 1 from the multiplier to get the number in the 10's position, and then adding enough to that number to make 9. For example, 9 × 5—take 1 from 5 to get the number of 10's (i.e., 4) then add enough to 4 to make 9 (i.e., 5), thus the answer is 45.	*The Multiplication/Division Relationship Rule*: to solve for 32 ÷ 4, think what number times 4 equals 32? The quotient times the divisor equals the dividend.

FIGURE 12.1. Basic facts rules.

- *Doubles*. These are the facts with identical addends (e.g., 4 + 4 or 6 + 6). Students are taught to associate the facts with a picture. For example, 4 + 4 could be the "spider fact" because a spider has 8 legs; 6 + 6 could be the "dozen eggs" fact.
- *Near doubles*. These are facts, such as 5 + 4 or 8 + 7, which are close to being doubles. Students are taught to relate these facts to their doubles. For example, if 4 + 4 = 8, then 5 + 4 must be 1 more. So 5 + 4 must equal 9.
- *9's*. The 9's facts (e.g., 9 + 4) follow a pattern. Count backward 1 from the small number and put the 1 in front. For example, for 9 + 4 the student would count backward 1 from 4 to get 3 and put the 1 in front (13). Students are taught to use this pattern.
- *10's*. The 10's facts are all the facts that sum to 10 (e.g., 7 + 3, 6 + 4). A 10 frame (see Figure 12.2) is used to teach these facts. Student are taught to remember the 10 frame.
- *Near 10's*. The near 10's (e.g., 9 + 2 or 8 + 4) are similar to the near doubles. Students are reminded to relate these facts to the appropriate 10's fact. For example, if 8 + 2 = 10, then 8 + 4 must be 12.

The key to this strategy is that all fact groups use a similar strategy for retrieval. Note that some facts are in more than one family. For example, 5 + 5 would be a double and a 10's fact. The strategy does not cover all addition facts (i.e., 7 + 5, 8 + 4, 8 + 5, 8 + 6). This small group of facts must be taught separately. They can be approached as "10's plus extras," or simply as extra facts and taught through memorization.

Multiplication

Multiplication facts are another important fact family. Once again, these facts are amenable to an approach that utilizes a chunking strategy. Wood and her colleagues (Wood & Frank, 2000; Wood, Frank, & Wacker, 1998) have developed and validated an effective strategy for teaching multiplication facts. The strategy divides the multiplication facts into six families: 0's, 1's, 2's, 5's, 9's, and pegwords (see Chapter 14 for more on

7 + 3 = 10

X	0
X	0
X	0
0	0
0	0

6 + 4 = 10

X	0
X	0
X	0
X	0
0	0

FIGURE 12.2. Ten-frame examples. A 10 frame is a container that has 10 separate receptacles. Manipulatives are used to illustrate the various combinations that add to 10.

pegwords). Facts are taught in this order. Note that the pegword facts are used for the 15 facts that do not fall into any of the first five families. Figure 12.3 shows the multiplication facts charts. Each fact family has its own strategy. In the case of the 2, 5, and 9 families, the strategy is combined with a graphic designed to serve as a mnemonic. Pegword facts use key words (i.e., the pegword) in sentences. Students are taught to associate the words with numbers (e.g., door = 4; gate = 8; dirty = 30; shoe = 2). Pegwords are put into sentences that combine illustrations to help the child learn and remember facts. For example, the problem 4 × 8 = 32 is represented by the sentence "Door on gate by dirty shoe" combined with a representation of these images.

Students are taught each fact family in turn. After all fact families have been mastered, students are taught to (1) scan a problem; (2) determine if one of the numbers is 0, 1, 2, 5, or 9; (3) if so, use the appropriate strategy; (4) if not, remember the pegword. When teaching, the strategy self-instructions are modeled and stressed (e.g., "This problem has a 9, so I can use my 9's strategy"). As always, instruction should stress that successful use of the strategy will help the student to get the right answer. Note that this strategy would follow naturally from the addition facts strategy previously discussed. The procedures are quite similar, and one fact family from the addition strategy, the

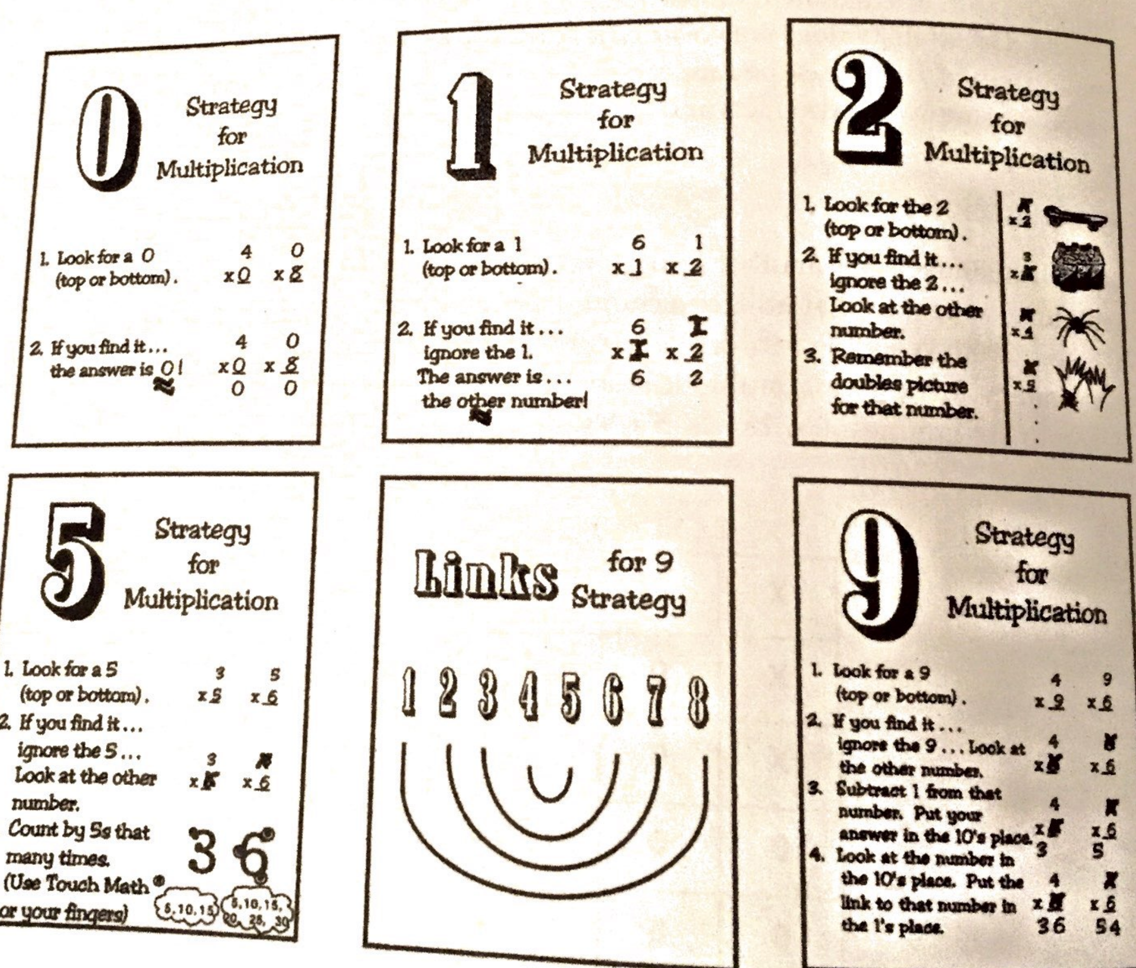


FIGURE 12.3. Multiplication strategy charts. From Wood and Frank (2000). Copyright 2000 by the Council for Exceptional Children. Reprinted by permission.

doubles, could be used for the 2's in the multiplication strategy. Commercial materials are available that use this approach to chunk strategies for multiplication facts (Liautaud & Rodreguez, 1999).

COMPUTATION STRATEGIES

Many students with LD struggle with computational procedures. Solving problems such as 34 × 7 or 456 + 895 requires students to remember necessary basic facts, the necessary steps, and how to order the steps. This can strain working memory for students with LD. As a result, they may forget the order of operation, regroup improperly, fail to regroup, or even develop improper algorithms. A number of strategies are useful for helping students remember and follow computational procedures. Figure 12.4 shows examples of computation strategies. These strategies typically consist of a mnemonic that serves to cue the student to remember and perform steps to correctly solve

DIVISION STRATEGY

Division. Remember *Does Mama Sell Biscuits Really?*:

Does (Divide)
Mama (Multiply)
Sell (Subtract)
Biscuits (Bring Down)
Really (Remainder)

___ Divide	___ Divide	___ Divide
___ Multiply	___ Multiply	___ Multiply
___ Subtract	___ Subtract	___ Subtract
___ Bring Down	___ Bring Down	___ Bring Down
___ Remainder	___ Remainder	___ Remainder
124 ÷ 7	66 ÷ 4	861 ÷ 9

MULTIPLICATION STRATEGY

Multiply. Remember MAMA:

Multiply the 1's column.
Across Do I need to go across to the 10's?
Multiply the bottom 1's digit with the top 10's digit.
Add any number that was carried in Step 2.

___ Multiply	___ Multiply	___ Multiply
___ Across	___ Across	___ Across
___ Multiply	___ Multiply	___ Multiply
___ Add	___ Add	___ Add
17 × 8	35 × 3	64 × 7

FIGURE 12.4. Examples of computation strategies.

problems. It is often useful to combine these types of strategies with self-monitoring. The self-monitoring serves to cue the student to perform all steps of the strategy in the correct order. As the student gains fluency with the strategy, the self-monitoring can be faded. Note that it is also possible to develop customized strategies for students. The process for creating customized computational strategies, developed by Dunlap and Dunlap (1989), was discussed in Chapter 7 (p. 114).

WORD-PROBLEM-SOLVING STRATEGIES

Many children with LD struggle with word problems. Word-problem solving requires students to apply basic facts and computational skills to novel situations. There are two components to word-problem solving (Jitendra, Hoff, & Beck, 1999). The first is *problem representation,* which entails translation of a problem from words into a meaningful combination of (typically) written symbols on paper (e.g., diagrams), actual objects (e.g., manipulatives), or mental imagery (Janvier, 1987). Constructing an effective representation is a prerequisite to understanding the quantitative relationships in a problem. The second is *problem solution,* which entails selection and application of appropriate mathematical operations based on the representation. Problem solution includes both solution planning and execution of appropriate mathematical operations. Research strongly suggests that two critical components for successful word-problem solving are explicit instruction in problem solving and the use of graphic representation of word problems such as diagrams (van Garderen, 2007).

Diagrams

Diagrams can be extremely helpful for students with MD and offer several benefits (van Garderen, 2006): (1) They reduce demands on working memory; (2) as visible reminders, diagrams allow for both teacher and student to monitor progress; (3) they are flexible and can be used for many different types of problems; and (4) they can be used as a part of a strategy. Note that using and creating effective diagrams is a skill that must be explicitly taught to students, especially those with LD (van Garderen, 2006). Simply telling students to "make a diagram" will not be effective. There are two basic types of diagrams: pictorial and schematic (Hegarty & Kozhenikov 1999). Pictorial diagrams depict the appearance of what is described in the problem; schematic diagrams depict the relations described in the problem (see Figure 12.5 for examples). The use of schematic diagrams is associated with successful problem solving, whereas the use of pictorial diagrams is associated with *unsuccessful* problem solving (van Garderen, 2007). Diagrams are an important part part of schema strategies (see below) that are effective for students with LD.

Four types of diagrams are commonly used (van Garderen, 2007; see Figure 12.6):

- Line diagrams—link points together with lines. They are useful to show how quantities may be combined in a problem.
- Matrices or tables—show relationships between two or more sets of information.

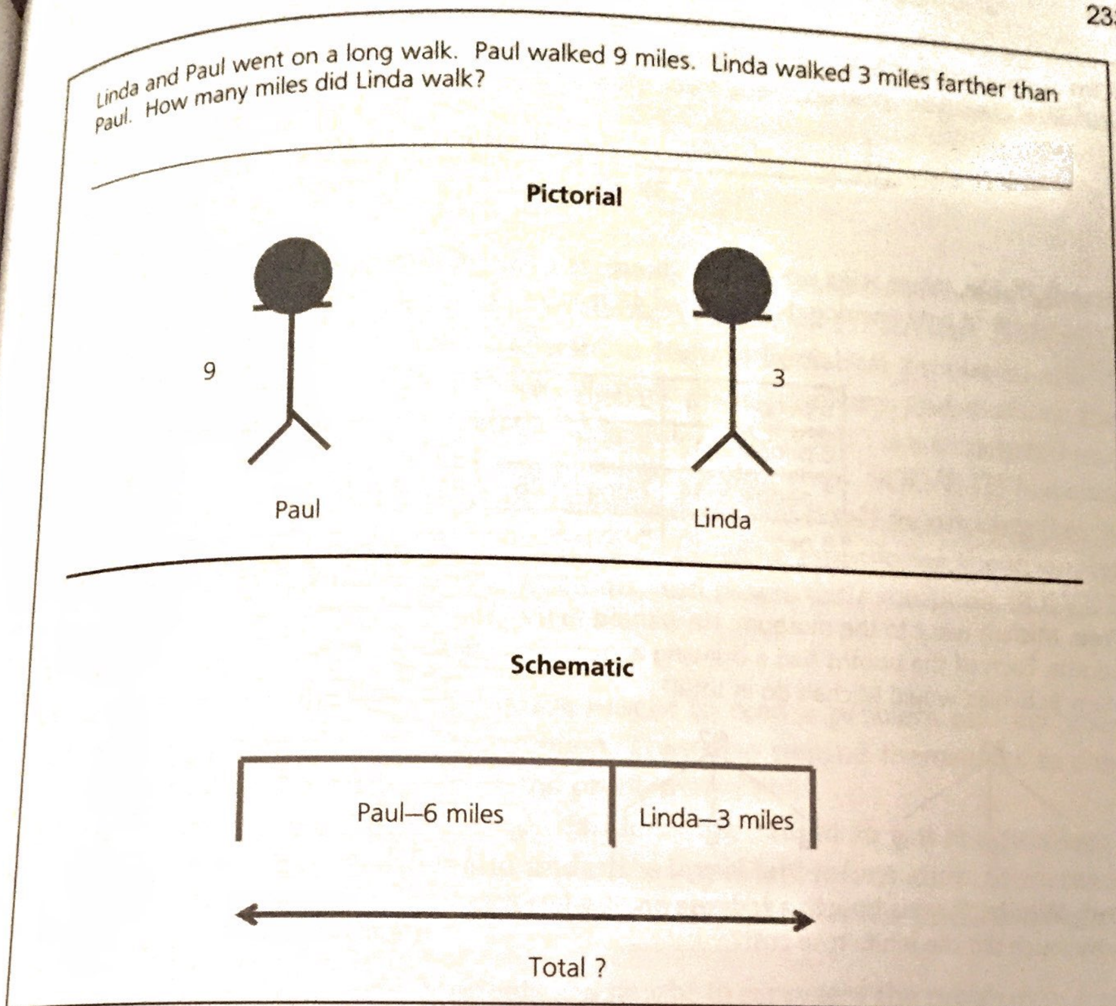


FIGURE 12.5. Examples of pictorial and schematic diagrams.

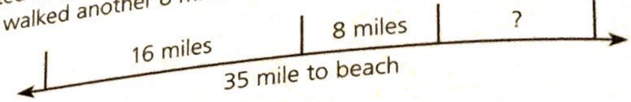
- Tree diagrams—show diverging or converging paths. They are useful in showing possible numbers of alternatives.
- Part–whole—show relationships between parts and the whole. They are useful for grouping components of a problem together.

Visualize

The visualize strategy (van Garderen, 2007) is designed to help students learn to create diagrams and use them to solve word problems. It uses the mnemonic RV-PCC along with self-instructions to guide students through the steps of the strategy. The steps in the visualize strategy follow:

1. *Read.* The students read the problem and check to see if they understand it.
2. *Visualize the problem.* Students draw a diagram of the problem and then assess it to determine whether it accurately shows how the parts of the problem are related. They rearrange the elements in the diagram if needed.

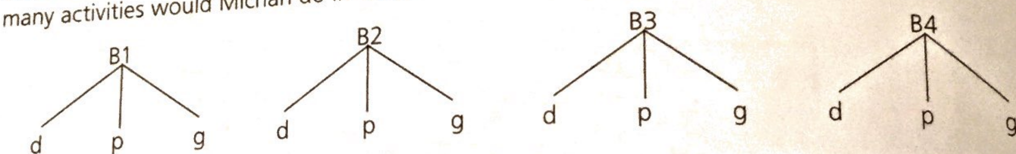
Line. Jordan wanted to get to the beach. The beach was 35 miles away. Jordan rode his bike for 16 miles. Then he walked another 8 miles. How far did Jordan need to go to reach the beach?



Matrix/Table. While Heidi was walking home, she saw many people walking their dogs. Each person walked only one dog. She counted 48 legs. How many people and how many dogs were there?

5 people =10	5 dogs = 20	30 legs
6 people = 12	6 dogs = 24	36 legs
7 people = 14	7 dogs = 28	42 legs
8 people = 16	8 dogs = 32	48 legs

Tree. Michah went to the museum. He wanted to try each activity at each booth. There were 4 booths. Each of the booths had a drawing activity, a physical activity, and guessing activity. How many activities would Michah do in total?



Part–Whole. Cynthia bought a red rose and a white rose for $26. She paid $11 for the red rose. How much did the white rose cost?

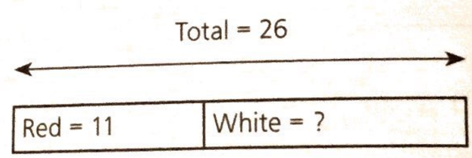


FIGURE 12.6. Examples of schematic diagrams.

3. *Plan how to solve the problem.* Here students determine what operations and how many steps are needed to solve the problem. After making the plan, students use their diagram to determine if the plan makes sense.
4. *Compute the answer.* Students solve the problem using the plan from the previous stage. They also check to see that their operations were done in the correct order.
5. *Check the answer.* Students check their answer to see if it makes sense. If not, they go back to their plan and ask for help if they need it.

During instruction, students also are taught important background information: (1) why they should use diagrams and how diagrams can be helpful; (2) rules to use when creating a diagram (i.e., there is no need to draw everything, drawings don't have to look realistic, and don't take to long making a diagram); (3) how to use symbols or graphic codes in diagrams (e.g., using the letter *b* for the number of beans); (4) how to use question marks to indicate unknowns; and (5) the major types of diagrams and how to use them.

SOLVE IT!

To solve a word problem, students must read the problem, decide what to do, solve the problem, and check that the answer is reasonable. The SOLVE IT! strategy (Montague, 2006) is designed to help students "understand the mathematical problems, analyze the information presented, develop logical plans to solve problems, and evaluate their solutions" (Montague, Warger, & Morgan, 2000, p. 111). SOLVE IT! is a structured strategy that uses explicit instruction to teach problem-solving steps. SOLVE IT! features a series of steps, each of which incorporates self-instructions and self-monitoring (i.e., *say, ask, check*). Students are taught to carefully read problems, paraphrase them, analyze the information, form a plan, solve the problems, and assess their solutions. SOLVE IT! involves the following steps:

1. *Read for understanding.* Students are taught to read a problem and ask themselves if they understood the problem. They also remind themselves to check for understanding as they work the problem.
2. *Put the problem in their own words.* Students are taught to put the problem in their own words and to find and underline important information. Students are taught to ask themselves what the question is and what the problem is asking for.
3. *Draw a picture or diagram.* Students are taught to represent the problem in a picture or diagram, and to check to see if their representation includes the important information in the problem.
4. *Make a plan to solve the problem.* Students identify the steps that will be involved in solving the problem and the operations (e.g., addition, subtraction, mulitiplication) that they will need. They write the symbols for the needed operations and check to see if the plan makes sense.
5. *Estimate the answer.* Students estimate what the final answer should look like by rounding numbers. They will check their final answer against this estimate.
6. *Do the problem.* Students compute the answer to the problem. They are taught to ask themselves to check their answer against their estimate, whether all operations were done in the correct order, and to ask whether the answer makes sense.
7. *Check the answer.* Finally, students check their computation and determine if it is correct and ask for help if needed.

Schema-Based Strategies

The wide variety of possible word problems is one factor that makes them more difficult for students. The ability to organize word problems into a small number of groups with common characteristics that can then be represented and solved simplifies the difficulty of word problems greatly. Schema-based strategies approach word problems from this perspective. Schemas are representations of word-problem structures. Schemas "capture both the patterns of relationships as well as their linkages to operations" (Marshall, 1995, p. 67). Thus, schema-based approaches allow students to both understand how to represent problems and to identify the correct operations for solving them (Jitendra, DiPipi, & Perron-Jones, 2002). An advantage of schema-based approaches is that when one piece of information is retrieved, other information that is linked to it will also be activated (Jitendra et al., 2002; Marshall, 1995). The most typical types of word-problem schemas in elementary and middle schools are *change, equalize, combine, compare, vary,* and *restate* (Riley, Greeno, & Heller, 1983; Van de Walle, 1998). Figures 12.7 and 12.8 show examples of problem types for each schema.

Jitendra and her colleagues (Jitendra, 2007; Jitendra & Hoff, 1996; Jitendra et al., 1998, 1999, 2000) have developed and validated a schema-based approach to word-problem solving. The strategy requires students to learn the types of schemas to mastery and to match each schema with the appropriate diagram (developed by Marshall, 1995). The diagram serves to remind students to record the important information and to cue the appropriate arithmetic operation. The steps in the strategy follow:

1. *Identify problem schemas*: Students are taught the types of schemas to mastery and how to differentiate between them through the use of several examples.
2. *Create an appropriate diagram*: Students are then taught how to appropriately diagram the different types of schemas (Figure 12.9). The diagrams serve as graphic organizers that help students organize and remember important information.
3. *Flag the missing element with a question mark*: The missing element, or the answer that the problem is requesting, is then flagged with a question mark. The question mark lets students know that they must use a mathematical operation to figure out the number to go in that box or circle (Figure 12.9).
4. *Apply the appropriate operation to solve the problem*: The type of schema and diagram will dictate the operation to be used. Students will need to be taught which operation goes with which type of schema and diagram (Figure 12.9).
5. *Ask if the answer made sense*: Once students have solved the problems, they are to check to see if the answers make sense (e.g., if the operation is addition, then the answer should be greater than both the addends). Students could use estimating to determine if their answers are reasonable.
6. *Check the work*: Students should be taught to "work the problem backward." Working a problem backward requires students to do the opposite operation to determine whether or not the answer is correct (e.g., subtraction → addition, multiplication → division).

Note that all of these steps must be taught to a high degree of mastery. Figure 12.9 shows examples of how a graphic organizer could be used with the different types of story problems. Commercial lesson plans to teach the schema approach to work-problem solving are now available (Jitendra, 2007).

Change

Results unknown

Dr. Gerber has 6 golf balls. Dr. Lloyd gave him 8 more. How many golf balls does Dr. Gerber have?

Sue has 21 cats. She gave 11 to John. How many cats does Sue have left?

Change unknown

Reese had 7 baseballs. Chris gave him some of her baseballs. Now Reese has 23 baseballs. How many baseballs did Chris give Reese?

John has 12 slices of pizza. He gave some pizza to Stan. Now John has 3 slices of pizza. How many slices did John give to Stan?

Start unknown

Torri had some hamburgers. Then Wendy gave her 7 hamburgers. Now Torri has 18 hamburgers. How many hamburgers did Torri have at the beginning?

Trevor had some cows. He gave 6 cows to Maci. Now Trevor has 22 cows. How many cows did Trevor have before he gave some to Maci?

Equalize

Mike has 7 dollars. Ron has 14 dollars. How many dollars does Mike need to have as many as Ron?

Javon has 25 trading cards. Fred has 11 trading cards. How many cards would Javon have to give away to have as many as Fred?

Combine

Total set unknown

Melody has 5 flowers. Emma has 9 flowers. How many flowers do they have in total?

Subset unknown

Emma and Leigh have 28 rabbits altogether. Leigh has 13 rabbits. How many rabbits does Emma have?

Compare

Difference unknown

Alex has 11 books. Joe has 5 books. Alex has how many more books than Joe?

Laura has 17 pens. Ross has 11 pens. Ross has how many fewer pens than Laura?

Compared quantity unknown

Andy has 4 computers. Matt has 8 more computers than Andy. How many computers does Matt have?

Laura has 23 books. Ross has 6 fewer books than Laura. How many books does Ross have?

Referent unknown

Jorge has 10 DVDs. He has 3 more DVDs than Cindy. How many DVDs does Cindy have?

Nirbhay has $17. He has $9 less than Alan. How many dollars does Alan have?

FIGURE 12.7. Addition and subtraction word-problem types.

Vary

Size of groups unknown

In Steve's basketball camp there are 5 balls for 25 players. How many players must share each ball?

Whole unknown

George worked picking up bottles for 6 days. He earned $54 for each day he picked up bottles. How much did George earn?

Compare

Referent unknown (compared is part of referent)

Mike and Ron bought some cherries. Mike bought 4 pounds of cherries. Ron bought one-third as many cherries as Mike. How many cherries did Ron buy?

Compared unknown (compared is part of referent)

Stan and Michalla both got speeding tickets. The amount that Stan had to pay was one-half the amount that Michalla had to pay. Stan had to pay $40. How much did Michalla have to pay?

Compared unknown (compared is multiple of referent)

John has 20 doughnuts. Evie has 4 times as many doughnuts as John. How many doughnuts does Evie have?

Restate

Susan and Lynette took a walk. Lynette walked half as far as Susan. If Susan walked 18 miles, how far did Lynette walk?

FIGURE 12.8. Multiplication and division word-problem types.

IMPLEMENTATION PLANS

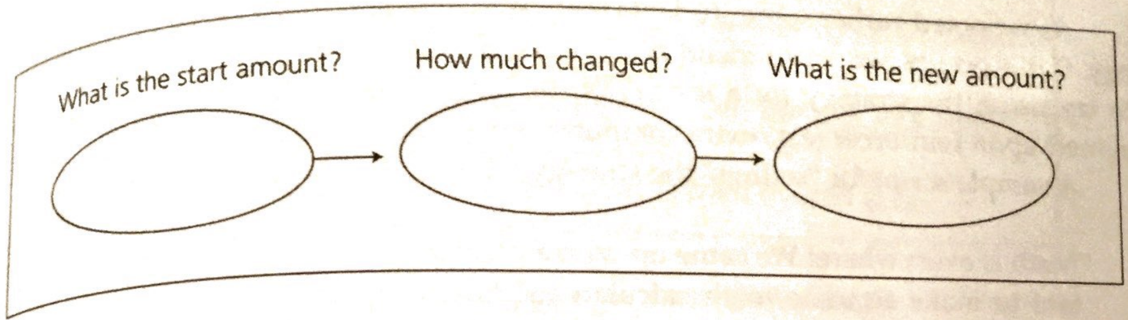
In this section, we provide partial examples of implementation plans for the math strategies previously discussed.

Stage 2 for Computation Strategies: Discussing the Strategy

This is the first stage in initiating the strategy when it is important to stress its relevance. During an initial conference the teacher discusses the students' current performance. It is also important for the teacher to stress the value of the strategy. Brainstorm with the students on situations wherein using this strategy or completing the given task accurately is important. For example, the following might be appropriate brainstorming

Change Problem

Frank has 8 seashells. Edward gave him 8 more. How many sea shells does Frank have now?



Vary Problem

In Steve's basketball camp there are 5 balls for 25 players. How many players must share each ball?

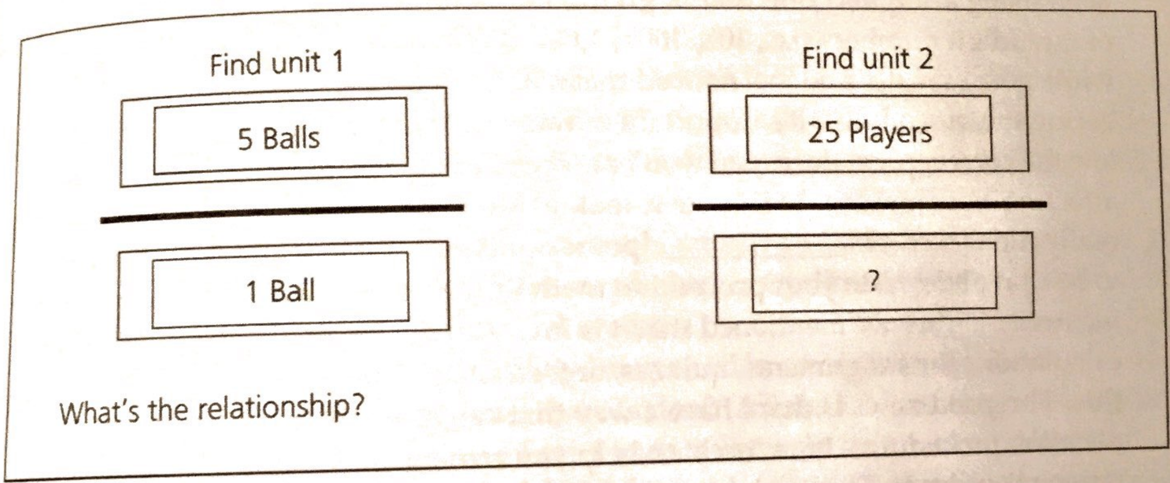


FIGURE 12.9. Examples of graphic organizers for word problems.

ideas in response to the question "When would it be important for you to make accurate math calculations?"

- Balancing your checkbook
- Trying to figure out if you have enough money to buy what you want
- Following a recipe
- Measuring anything
- Building a house
- Fixing your car

- Planning next year's crops
- For a test
- For an assignment

As we noted earlier some students may be unsure about, or reluctant to try, a strategy. If this occurs, we recommend the use of a behavioral contract. The student agrees to try using the strategy for a set period of time. In return, the student receives an agreed-upon reinforcer (e.g., extra computer time).

A sample script for "selling" the strategy follows:

> "Math is everywhere! We came up with a lot of instances where it would be important to make accurate math calculations. We decided that it is really important when we are balancing our checkbook, and seeing if we have enough money to buy that new CD we want. Math is a very important part of our lives, whether we like it or not. Sometimes it's difficult to remember all of the different steps in various math calculations, or our 'computational procedures.' One procedure we have been using a lot, and one that is giving many of us difficulties, is multiplication of multidigit numbers (i.e., 10's, 100's, 1,000's). We have been working on this for a while now in math, and I've noticed that many of you are having difficulty remembering the procedures. It's important to remember the procedures; if you don't follow the correct procedure, you won't get the correct answer. This problem has been affecting your grades. Let's take a look at some of your recent quiz scores. You really did a nice job ____________ [point out the positives]. However, there seems to be a breakdown in your procedural methods because you are getting quite a few incorrect. Earlier we mentioned that it is important to be able to make correct math calculations for assignments, quizzes, or tests; it appears that we could improve on this. The good news is that I have a way that can really help you with your multiplication procedures. It's a 'trick' to help you remember the multiplication computation procedures. This trick is a strategy called *MAMA*. Just like your mom's help, this strategy will help you! It should make multiplication easier for you, and your grades will improve."

In this stage we also introduce the strategy steps and any prompts that will be given (Figure 12.10).

Stage 3 for SOLVE IT!: Modeling the Strategy

The teacher will need to use a think-aloud to demonstrate the use of the strategy. Here is an example of a think-aloud for SOLVE IT!

> "OK, I am doing my math homework. Things are going just fine; I know how to do these subtraction problems with regrouping when they are written out in standard form. OK, but now I am up to the last section and they are *story problems*! Story problems always give me trouble! Ugh, what can I do? Oh, yeah, the other day we talked about a trick that could help us with our story problems; it was called the

MAMA

Step 1: Multiply the 1's column.

Step 2: Across—do I need to go across to the 10's?

Step 3: Multiply the bottom 1's digit with the top 10's digit.

Step 4: Add any number that was carried in Step 2.

FIGURE 12.10. Prompt sheet for the MAMA strategy.

'SOLVE IT!' strategy. I'll try it now. OK, first I need to *Read* for understanding. Being able to read the problem is important, but it's even more important that I understand what I've read. If I don't understand it, I'll never be able to pull out all of the necessary information to solve it.

"At each step in the SOLVE IT! strategy I am supposed to follow three steps: (1) Say, (2) Ask, and (3) Check. So, at this step I need to *say*: 'Read the problem. If I don't understand, read it again.' OK, so I'll read the problem. The problem says: 'Taylon has 427 head of cattle on his ranch. Sydni has 605 head of cattle on her ranch. How many more cattle does Sydni have than Taylon?' All right, I read the problem, now what? I need to *ask* myself, 'Have I read and understood the problem?' Well, I read the problem; did I understand it? I think so. Now I can just go to the next step and check for understanding as I solve the problem. I double-check to make sure I understand the problem while I'm attempting to solve it. Now what do I need to do? The next step is to *Paraphrase* the problem in my own words, so I *say*: 'Underline the important information. Put the problem in my own words.' Well, I know that the number of cattle is important. I know that because the question asks me 'How many more?' Numbers in story problems are usually important.

"OK, now I need to put the problem in my own words. Sydni has more cows than Taylon. She has 605, and he has 427. How many more does she have? This is going pretty smooth; now I need to *ask*: 'Have I underlined the important information?' Yes, I underlined the number of cattle each rancher had. Those are the numbers I need to work with. Good. Now, 'What is the question? What am I looking for?' The question is, 'How many more head of cattle does Sydni have than Taylon?' So, I'm looking for how many MORE. OK, now I need to *check* that the information goes with the question. So, the question asks how many head of cattle? That's the information that I underlined; that's right then. It's helpful to do these checks to make sure I'm not messing up or leaving out important steps along the way.

"What's next? *Visualize* a picture of a diagram. I know what a diagram is. That's like a graph or a web. OK, I need to *say*: 'Make a drawing or diagram.' All right, I've drawn my graphic representation. Now I need to *ask*: 'Does the picture fit the problem?' Well, it shows Sydni's ranch with her cattle, and Taylon's ranch with his cattle; yes, it does fit the problem! Let's keep going! Next, I need to *Hypothesize* a plan to solve the problem. What does *hypothesize* mean? I know in science it means to make an educated guess, so I think this step means I need to make an educated guess at a plan to solve this problem. I'm now supposed to *say*: 'Decide how many steps and operations are needed. Write the operation symbols (+, −, ×, ÷). This problem has only one step.' I need to find the difference between Taylon's herd and Sydni's herd. I already know how many cattle each of them has; all I need to do is find the difference. I know that *difference* almost always means subtraction, so I will write down a −. Now, I need to *ask*: If I do subtraction, what will I get? Well, I will get a number smaller than my largest number, but I don't know what that will be until I actually solve the problem.

"OK, so I know that I only need to perform one operation, and that operation is subtraction. This sounds like a reasonable plan. Now, I need to *check* that the plan makes sense. Well, I already said that it sounds reasonable. That means I think my plan makes sense. How will I know? Let's see, what's my next step? My next step is to *Estimate* and predict my answer. That will help me to make sure my plan makes sense. All right, on to the next step; I need to *say*: 'Round the numbers, do the problem in my head, and write the estimate.' OK, so if I round the larger number, 605, I get 600. I round the larger number first because I know with subtraction the larger number is always on top. All right, now the other number, 427, and I get 400. Now I need to do the problem in my head (600 − 400 = 200) and write the estimate, so I'll write down *200*. Now, I need to *ask*: 'Did I round up and down?' Well, I only rounded down, but that's because I know when I round, if a number is less than 5 I round down, but if a number is 5 or greater I round up. In both of the numbers that I was rounding, the second digit was less than 5, so I rounded down. I still need to *ask*: 'Did I write the estimate?' Yes, I wrote it on the board.

"Now, I need to *check* that I used important information. Yes, I did! I used the information that I underlined. I didn't even realize it at first, but having the numbers underlined really helped me find the correct information. Wow, this is going very well; this strategy is really helping me with my story problem. What's next? *Compute*, or do the arithmetic. I'm finally ready to solve the problem. I feel very confident that I'm going to be able to SOLVE IT! Let's see, 605 − 427 = ? OK, I got it, 178. Now, I need to *ask*: 'How does my answer compare with my estimate? Does my answer make sense? Are the decimals or money signs in the right place?' Well, my answer is very close to my estimate; if I rounded my answer, I would get my estimate! I think I did it—my answer definitely makes sense. There are no decimals or money symbols, so I don't need to worry about that. Almost done, only one more step! I need to *Check* to make sure everything is right. So, I *say*: 'Check the computation.' OK, I used the correct operation, and my estimate is close to my answer, but I'm not sure if it's completely accurate. How can I make sure my answer is correct? Well, we usually check our work by performing the opposite operation; even though this is a story problem this should still work—all I'm doing is subtraction! OK, so I can do addition to check. I will add my answer to the number I subtracted, and I should get the larger number. I already know how to do this! All right, so 178 + 427 = 605; yes, I got it right!

"Next, 'Have I checked every step? Have I checked the computation? Is my answer right?' Yes, I checked every step (I only had one), I checked my computation by doing the opposite operation, and I determined that I had the correct answer! Finally, I need to *check* that everything is right. If not, go back. Then ask for help if I need it. Well, I already asked myself these questions, and I determined that I got the right answer, so I don't need to go back or ask for help. I did it! I did it, and I know I got it all right. Wow, when I try and make sure I use the appropriate strategy, story problems aren't so bad! That was actually kind of fun!"

Stage 5 for Schema-Based Strategies: Supporting the Strategy

Students need to automatically recognize the various types of schemas used. They also need to be able to identify the features of the semantic relations in the problem and check the salient element of the chosen problem schema. Based on Jitendra's (Jitendra et al., 1998) procedures, scaffolding should include the following:

1. The teacher models collaboratively with students. Students help identify critical elements or constraints of the word problems.

- Students practice identifying the different problem types in the story situations (e.g., Change, Group, and Compare).
- Students translate the information (i.e., read and understand the word problem).
- Students map the features of the situation onto the schema diagrams (Figure 12.9).

2. The teacher reviews the problem schemas through collaborative modeling. Any student misconceptions should be addressed immediately, and explicit feedback provided along with additional modeling.

- Equations, instead of word problems, are presented; however, the teacher still *reads* a word problem to the students.
- The teacher should use a facilitative questioning procedure for students to identify the semantic features of the problem.
- The teacher then demonstrates how critical elements of the specific problem are translated and mapped on the schema diagrams (Figure 12.9).
- The missing element is flagged with a question mark.

3. Instruction is given on the second strategy step. During this stage, students should be given explicit feedback on their ability to synthesize the steps of the strategy. The teacher should use guided practice and modeling, and provide immediate corrective feedback at each step in the instructional process.

- The teacher explains how to find the total amount in the word problem by focusing on the specific information in the problem. For example:
  - *Change problem*: Students will need to determine if the problem ends with more or less than the beginning amount.
    - ending state = total when the change results in an increase
    - beginning state = total when the change results in a decrease
  - *Compare problem*: Students compare the value of the referent and compare objects to determine the greater quantity, or the total.
  - Students must read the comparison or difference statement in the word problem.
  - Students are taught a generalizable rule based on the part–whole concept for determining the operation to be used. They must examine the part of the situation that is unknown and whether it represents the whole or the part to be solved.
    - If the whole is *not* known, then add to find total.
    - If the whole *is* known, then subtract to the other (part) amount.

Initially, students should be given only one type of problem; after they are able to successfully use the strategy with that type of problem, others should be introduced one at a time.

Stage 5 for Basic Multiplication Strategies: Supporting the Strategy

Scaffolding is important in this stage. With scaffolding it is possible to gradually transfer strategy performance from teacher to student. Students need to be given adequate time and support to master the strategy.

Content Scaffolding

Students are given simple multiplication problems. The teacher and students then go over the problems and the use of the different strategies. Together they determine which type of multiplication strategy is appropriate, and they solve the problem together. The teacher directs the process and the students provide answers to teacher-directed questions (i.e., "What type of multiplication strategy does this problem represent? What do we need to remember to solve this problem?").

Task Scaffolding

Students are taught one multiplication strategy at a time, first the "0 strategy" for multiplication to mastery, then each additional strategy, as listed in Figure 12.3.

During collaborative practice the teacher prompts students to use their various multiplication strategies to solve problems. The teacher demonstrates the use of the various multiplication strategies though modeling. In subsequent lessons the teacher asks students to identify the strategy necessary to solve the problem and asks how they knew which strategy to use (the teacher directs the process). Finally, students are given a set of multiplication problems and expected to choose the appropriate multiplication strategy and answer the questions correctly.

Material Scaffolding

Students are given prompt cards with multiplication strategies (Figure 12.3) to use with their independent math work. Initially, the prompts serve as a guide to remind them of various multiplication strategies, and the cues that go with each. Students are provided with multiple opportunities to practice using their multiplication strategies, until they are able to do so independently and to successfully solve basic multiplication problems.

Stage 6 for Computational Strategies: Independent Performance

At this stage the student is ready to use the strategy independently. The teacher's main task is to monitor the student's performance and check on proper and consistent use of the strategy. It is important to keep in mind that our main goal is improved academic performance. Teachers must evaluate whether or not the strategy is being used, if it is being generalized to other appropriate situations, and whether or not academic performance has improved. Students will not always generalize strategies to appropriate situations; they will need to be prompted and encouraged to do so.

To promote generalization, students are encouraged to use the strategy in other content areas where they are required to compute mathematical calculations. All team teachers should be informed about the use of the strategy, the language, the prompts, and what is required at each step. All team teachers are given a wall chart to hang up in their room as a reminder for students to use the computational strategies when appropriate.

Students should be assessed regularly, using independent work, quizzes, and tests, specifically in math class. These scores are recorded and tracked for trends in progress. The goal is improved academic performance, and if performance isn't improving, a reteaching of the strategies may be necessary. Once students are successful in using the strategies, their performance should be monitored periodically. Even though they have reached the independent performance stage, they should be monitored to ensure proper use of the strategy. If students deviate from the given computation strategies, performance should be evaluated and action taken only if performance is no longer improving.

FINAL THOUGHTS

There is good evidence that strategies can be effective for all the levels of mathematics included in this chapter. In fact, because it is often highly procedural and rule based, mathematics is a natural area for strategy instruction. Still, there are some areas (e.g., algebra, geometry) that have not been well studied or have not been studied with students with LD. In these cases, teachers would need to construct their own strategies using a task analysis as a guide. An accurate task breakdown, combined with the SRSD model, should be helpful when there are no existing strategies.