DYNAMICS

COMMON NOMENCLATURE

t = time
s = position
v = velocity
a = acceleration
a_n = normal acceleration
a_t = tangential acceleration
θ = angle
ω = angular velocity
α = angular acceleration
Ω = angular velocity of x,y,z reference axis
$\dot{\Omega}$ = angular acceleration of reference axis
$r_{A/B}$ = relative position of "A" with respect to "B"
$v_{A/B}$ = relative velocity of "A" with respect to "B"
$a_{A/B}$ = relative acceleration of "A" with respect to "B"

PARTICLE KINEMATICS

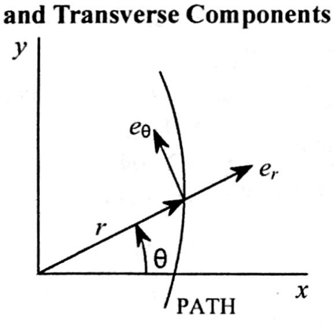
Kinematics is the study of motion without consideration of the mass of, or the forces acting on, a system. For particle motion, let $\boldsymbol{r}(t)$ be the position vector of the particle in an inertial reference frame. The velocity and acceleration of the particle are defined, respectively, as

$\boldsymbol{v} = d\boldsymbol{r}/dt$
$\boldsymbol{a} = d\boldsymbol{v}/dt$, where
$\boldsymbol{v}$ = the instantaneous velocity
$\boldsymbol{a}$ = the instantaneous acceleration
t = time

Cartesian Coordinates

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$
$$\boldsymbol{v} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}$$
$$\boldsymbol{a} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}, \text{ where}$$
$$\dot{x} = dx/dt = v_x, \text{ etc.}$$
$$\ddot{x} = d^2x/dt^2 = a_x, \text{ etc.}$$

Radial and Transverse Components for Planar Motion



Unit vectors $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_r$ are, respectively, normal to and collinear with the position vector $\boldsymbol{r}$. Thus:

$$\boldsymbol{r} = r\boldsymbol{e}_r$$
$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta, \text{ where}$$

r = the radial distance
θ = the angle between the x axis and $\boldsymbol{e}_r$
$\dot{r} = dr/dt$, etc., $\ddot{r} = d^2r/dt^2$, etc.

♦ Particle Rectilinear Motion

Variable a	*Constant a = a_c*
$a = \dfrac{dv}{dt}$	$v = v_0 + a_c t$
$v = \dfrac{ds}{dt}$	$s = s_0 + v_0 t + \frac{1}{2}a_c t^2$
$a\,ds = v\,dv$	$v^2 = v_0^2 + 2a_c(s - s_0)$

♦ Particle Curvilinear Motion

x,y,z Coordinates		*r,θ,z Coordinates*	
$v_x = \dot{x}$	$a_x = \ddot{x}$	$v_r = \dot{r}$	$a_r = \ddot{r} - r\dot{\theta}^2$
$v_y = \dot{y}$	$a_y = \ddot{y}$	$v_\theta = r\dot{\theta}$	$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$v_z = \dot{z}$	$a_z = \ddot{z}$	$v_z = \dot{z}$	$a_z = \ddot{z}$

n, t, b Coordinates

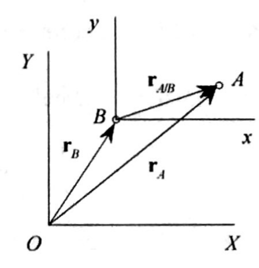
$v = \dot{s}$	$a_t = \dot{v} = \dfrac{dv}{dt}$	
	$a_n = \dfrac{v^2}{\rho}$	$\rho = \dfrac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left\lvert \dfrac{d^2y}{dx^2} \right\rvert}$

♦ Relative Motion

$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$ $\quad \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ $\quad \mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$

Translating Axes x-y

The equations that relate the absolute and relative position, velocity, and acceleration vectors of two particles A and B, in plane motion may be written as

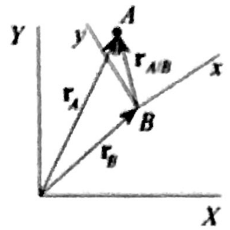


$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$
$$\mathbf{v}_A = \mathbf{v}_B + \omega \times \mathbf{r}_{A/B} = \mathbf{v}_B + \mathbf{v}_{A/B}$$
$$\mathbf{a}_A = \mathbf{a}_B + \alpha \times \mathbf{r}_{A/B} + \omega \times (\omega \times \mathbf{r}_{A/B}) = \mathbf{a}_B + \mathbf{a}_{A/B}$$

where ω and α are the absolute angular velocity and absolute angular acceleration of the relative position vector $\mathbf{r}_{A/B}$, respectively.

♦ Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.

Rotating Axis



$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$

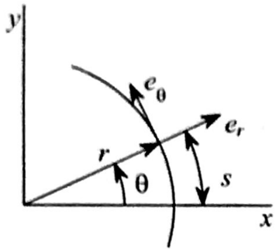
$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{A/B}$

$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{A/B} + \mathbf{a}_{A/B}$

where ω and α are, respectively, the total angular velocity and acceleration of the relative position vector $\boldsymbol{r}_{A/B}$.

Plane Circular Motion

A special case of transverse and radial components is for constant radius rotation about the origin, or plane circular motion.



Here the vector quantities are defined as

$\boldsymbol{r} = r\boldsymbol{e}_r$

$\boldsymbol{v} = r\omega\boldsymbol{e}_\theta$

$\boldsymbol{a} = (-r\omega^2)\boldsymbol{e}_r + r\alpha\boldsymbol{e}_\theta$, where

r = the radius of the circle

θ = the angle between the x and $\boldsymbol{e}_r$ axes

The magnitudes of the angular velocity and acceleration, respectively, are defined as

$\omega = \dot{\theta}$

$\alpha = \dot{\omega} = \ddot{\theta}$

Arc length, tangential velocity, and tangential acceleration, respectively, are

$s = r\theta$

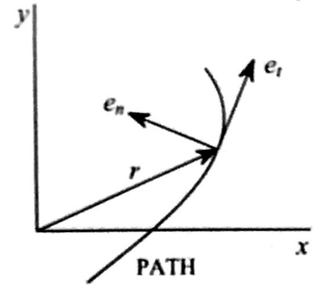
$v_\theta = r\omega$

$a_\theta = r\alpha$

The normal acceleration is given by

$a_r = -r\omega^2$ (towards the center of the circle)

Normal and Tangential Components



Unit vectors $\boldsymbol{e}_t$ and $\boldsymbol{e}_n$ are, respectively, tangent and normal to the path with $\boldsymbol{e}_n$ pointing to the center of curvature. Thus

$\boldsymbol{v} = v(t)\boldsymbol{e}_t$

$\boldsymbol{a} = a(t)\boldsymbol{e}_t + (v_t^2/\rho)\boldsymbol{e}_n$, where

ρ = instantaneous radius of curvature

Constant Acceleration

The equations for the velocity and displacement when acceleration is a constant are given as

$a(t) = a_0$

$v(t) = a_0(t - t_0) + v_0$

$s(t) = a_0(t - t_0)^2/2 + v_0(t - t_0) + s_0$, where

s = distance along the line of travel

s_0 = displacement at time t_0

v = velocity along the direction of travel

v_0 = velocity at time t_0

a_0 = constant acceleration

t = time

t_0 = some initial time

For a free-falling body, $a_0 = -g$ (downward).
An additional equation for velocity as a function of position may be written as

$$v^2 = v_0^2 + 2a_0(s - s_0)$$

For constant angular acceleration, the equations for angular velocity and displacement are

$\alpha(t) = \alpha_0$

$\omega(t) = \alpha_0(t - t_0) + \omega_0$

$\theta(t) = \alpha_0(t - t_0)^2/2 + \omega_0(t - t_0) + \theta_0$, where

θ = angular displacement

θ_0 = angular displacement at time t_0

ω = angular velocity

ω_0 = angular velocity at time t_0

α_0 = constant angular acceleration

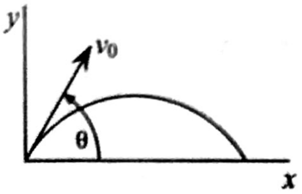
t = time

t_0 = some initial time

An additional equation for angular velocity as a function of angular position may be written as

$$\omega^2 = \omega_0^2 + 2\alpha_0(\theta - \theta_0)$$

Projectile Motion



The equations for common projectile motion may be obtained from the constant acceleration equations as

$a_x = 0$

$v_x = v_0 \cos(\theta)$

$x = v_0 \cos(\theta) t + x_0$

$a_y = -g$

$v_y = -gt + v_0 \sin(\theta)$

$y = -gt^2/2 + v_0 \sin(\theta) t + y_0$

Non-constant Acceleration

When non-constant acceleration, $a(t)$, is considered, the equations for the velocity and displacement may be obtained from

$$v(t) = \int_{t_0}^{t} a(\tau) d\tau + v_{t_0}$$

$$s(t) = \int_{t_0}^{t} v(\tau) d\tau + s_{t_0}$$

For variable angular acceleration

$$\omega(t) = \int_{t_0}^{t} \alpha(\tau) d\tau + \omega_{t_0}$$

$$\theta(t) = \int_{t_0}^{t} \omega(\tau) d\tau + \theta_{t_0}$$

where τ is the variable of integration

CONCEPT OF WEIGHT

$W = mg$, where

W = weight, N (lbf)

m = mass, kg (lbf-sec²/ft)

g = local acceleration of gravity, m/s² (ft/sec²)

PARTICLE KINETICS

Newton's second law for a particle is

$\Sigma F = d(mv)/dt$, where

ΣF = the sum of the applied forces acting on the particle

m = the mass of the particle

v = the velocity of the particle

For constant mass,

$\Sigma F = m\, dv/dt = ma$

One-Dimensional Motion of a Particle (Constant Mass)

When motion exists only in a single dimension then, without loss of generality, it may be assumed to be in the x direction, and

$a_x = F_x/m$, where

F_x = the resultant of the applied forces, which in general can depend on t, x, and v_x.

If F_x only depends on t, then

$$a_x(t) = F_x(t)/m$$

$$v_x(t) = \int_{t_0}^{t} a_x(\tau) d\tau + v_{xt_0}$$

$$x(t) = \int_{t_0}^{t} v_x(\tau) d\tau + x_{t_0}$$

where τ is the variable of integration

If the force is constant (i.e., independent of time, displacement, and velocity) then

$$a_x = F_x/m$$

$$v_x = a_x(t - t_0) + v_{xt_0}$$

$$x = a_x(t - t_0)^2/2 + v_{xt_0}(t - t_0) + x_{t_0}$$

Normal and Tangential Kinetics for Planar Problems

When working with normal and tangential directions, the scalar equations may be written as

$\Sigma F_t = ma_t = m dv_t/dt$ and

$\Sigma F_n = ma_n = m(v_t^2/\rho)$

♦ Kinetic Energy

Particle	$T = \frac{1}{2}mv^2$
Rigid Body (*Plane Motion*)	$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

♦ Potential Energy

$V = V_g + V_e$, where $V_g = \pm Wy$, $V_e = + 1/2\ ks^2$

The work done by an external agent in the presence of a conservative field is termed the change in potential energy.

Potential Energy in Gravity Field

$V_g = mgh$, where

h = the elevation above some specified datum.

Elastic Potential Energy

For a linear elastic spring with modulus, stiffness, or spring constant, the force in the spring is

$F_s = k\, s$, where

s = the change in length of the spring from the undeformed length of the spring.

In changing the deformation in the spring from position s_1 to s_2, the change in the potential energy stored in the spring is

$$V_2 - V_1 = k(s_2^2 - s_1^2)/2$$

♦ Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.

◆ **Work**

Work U is defined as

$$U = \int \mathbf{F} \cdot d\mathbf{r}$$

Variable force	$U_F = \int F \cos\theta \, ds$
Constant force	$U_F = (F_c \cos\theta)\Delta s$
Weight	$U_W = -W\Delta y$
Spring	$U_s = -\left(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2\right)$

where s_1 and s_2 are two different positions of the applied force end of the spring with $|s_2|>|s_1|$.

Couple moment $U_M = M\Delta\theta$

◆ **Power and Efficiency**

$$P = \frac{dU}{dt} = \mathbf{F}\cdot\mathbf{v} \qquad \varepsilon = \frac{P_{out}}{P_{in}} = \frac{U_{out}}{U_{in}}$$

Principle of Work and Energy

If T_i and V_i are, respectively, the kinetic and potential energy of a particle at state i, then for conservative systems (no energy dissipation or gain), the law of conservation of energy is

$$T_2 + V_2 = T_1 + V_1$$

If nonconservative forces are present, then the work done by these forces must be accounted for. Hence

$$T_2 + V_2 = T_1 + V_1 + U_{1\to2}, \text{ where}$$

$U_{1\to2}$ = the work done by the nonconservative forces in moving between state 1 and state 2. Care must be exercised during computations to correctly compute the algebraic sign of the work term. If the forces serve to increase the energy of the system, $U_{1\to2}$ is positive. If the forces, such as friction, serve to dissipate energy, $U_{1\to2}$ is negative.

Impulse and Momentum

Linear

Assuming constant mass, the equation of motion of a particle may be written as

$$m\,dv/dt = F$$

$$m\,dv = F\,dt$$

For a system of particles, by integrating and summing over the number of particles, this may be expanded to

$$\Sigma m_i (\mathbf{v}_i)_{t_2} = \Sigma m_i (\mathbf{v}_i)_{t_1} + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i \, dt$$

The term on the left side of the equation is the linear momentum of a system of particles at time t_2. The first term on the right side of the equation is the linear momentum of a system of particles at time t_1. The second term on the right side of the equation is the impulse of the force F from time t_1 to t_2. It should be noted that the above equation is a vector equation. Component scalar equations may be obtained by considering the momentum and force in a set of orthogonal directions.

Angular Momentum or Moment of Momentum

The angular momentum or the moment of momentum about point 0 for a particle is defined as

$$\mathbf{H}_0 = \mathbf{r} \times m\mathbf{v}, \text{ or}$$

$$\mathbf{H}_0 = I_0 \omega$$

Taking the time derivative of the above, the equation of motion may be written as

$$\dot{\mathbf{H}}_0 = d(I_0\omega)/dt = \mathbf{M}, \text{ where}$$

M is the moment applied to the particle. Now by integrating and summing over a system of any number of particles, this may be expanded to

$$\Sigma(\mathbf{H}_{0i})_{t_2} = \Sigma(\mathbf{H}_{0i})_{t_1} + \Sigma \int_{t_1}^{t_2} M_{0i}\,dt$$

The term on the left side of the equation is the angular momentum of a system of particles at time t_2. The first term on the right side of the equation is the angular momentum of a system of particles at time t_1. The second term on the right side of the equation is the angular impulse of the moment M from time t_1 to t_2.

Impact

During an impact, momentum is conserved while energy may or may not be conserved. For direct central impact with no external forces

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2, \text{ where}$$

m_1, m_2 = the masses of the two bodies

v_1, v_2 = the velocities of the bodies just before impact

v'_1, v'_2 = the velocities of the bodies just after impact

For impacts, the relative velocity expression is

$$e = \frac{(v'_2)_n - (v'_1)_n}{(v_1)_n - (v_2)_n}, \text{ where}$$

e = coefficient of restitution

$(v_i)_n$ = the velocity normal to the plane of impact just **before** impact

$(v'_i)_n$ = the velocity normal to the plane of impact just **after** impact

The value of e is such that

$0 \le e \le 1$, with limiting values

$e = 1$, perfectly elastic (energy conserved)

$e = 0$, perfectly plastic (no rebound)

Knowing the value of e, the velocities after the impact are given as

$$(v'_1)_n = \frac{m_2(v_2)_n(1+e) + (m_1 - em_2)(v_1)_n}{m_1 + m_2}$$

$$(v'_2)_n = \frac{m_1(v_1)_n(1+e) - (em_1 - m_2)(v_2)_n}{m_1 + m_2}$$

◆ Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.

Friction

The Laws of Friction are

1. The total friction force F that can be developed is independent of the area of contact.
2. The total friction force F that can be developed is proportional to the normal force N.
3. For low velocities of sliding, the total frictional force that can be developed is practically independent of the velocity, although experiments show that the force F necessary to initiate slip is greater than that necessary to maintain the motion.

The formula expressing the Laws of Friction is

$$F \le \mu N, \text{ where}$$

μ = the coefficient of friction.

In general

$F < \mu_s N$, no slip occurring

$F = \mu_s N$, at the point of impending slip

$F = \mu_k N$, when slip is occurring

Here,

μ_s = the coefficient of static friction

μ_k = the coefficient of kinetic friction

PLANE MOTION OF A RIGID BODY

Kinematics of a Rigid Body

Rigid Body Rotation

For rigid body rotation θ

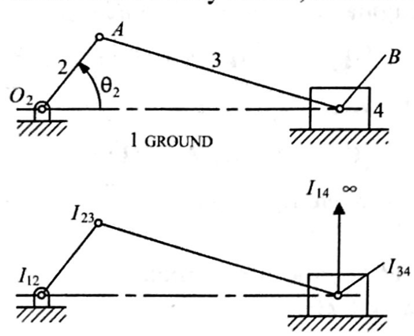
$$\omega = d\theta/dt$$

$$\alpha = d\omega/dt$$

$$\alpha d\theta = \omega d\omega$$

Instantaneous Center of Rotation (Instant Centers)

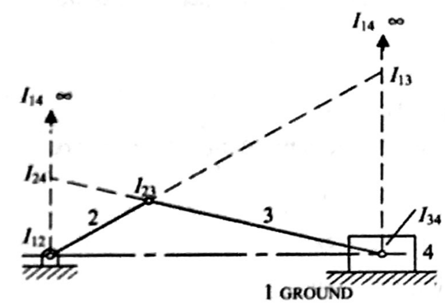
An instantaneous center of rotation (instant center) is a point, common to two bodies, at which each has the same velocity (magnitude and direction) at a given instant. It is also a point in space about which a body rotates, instantaneously.



The figure shows a fourbar slider-crank. Link 2 (the crank) rotates about the fixed center, O_2. Link 3 couples the crank to the slider (link 4), which slides against ground (link 1). Using the definition of an instant center (IC), we see that the pins at O_2, A, and B are ICs that are designated I_{12}, I_{23}, and I_{34}. The easily observable IC is I_{14}, which is located at infinity with its direction perpendicular to the interface between links 1 and 4 (the direction of sliding). To locate the remaining two ICs (for a fourbar) we must make use of Kennedy's rule.

***Kennedy's Rule:** When three bodies move relative to one another they have three instantaneous centers, all of which lie on the same straight line.*

To apply this rule to the slider-crank mechanism, consider links 1, 2, and 3 whose ICs are I_{12}, I_{23}, and I_{13}, all of which lie on a straight line. Consider also links 1, 3, and 4 whose ICs are I_{13}, I_{34}, and I_{14}, all of which lie on a straight line. Extending the line through I_{12} and I_{23} and the line through I_{34} and I_{14} to their intersection locates I_{13}, which is common to the two groups of links that were considered.



Similarly, if body groups 1, 2, 4 and 2, 3, 4 are considered, a line drawn through known ICs I_{12} and I_{14} to the intersection of a line drawn through known ICs I_{23} and I_{34} locates I_{24}.

The number of ICs, c, for a given mechanism is related to the number of links, n, by

$$c = \frac{n(n-1)}{2}$$

Kinetics of a Rigid Body

In general, Newton's second law for a rigid body, with constant mass and mass moment of inertia, in plane motion may be written in vector form as

$$\Sigma \mathbf{F} = m\mathbf{a}_c$$

$$\Sigma \mathbf{M}_c = I_c \alpha$$

$$\Sigma \mathbf{M}_p = I_c \alpha + \boldsymbol{\rho}_{pc} \times m\mathbf{a}_c, \text{ where}$$

$\mathbf{F}$ are forces and $\mathbf{a}_c$ is the acceleration of the body's mass center both in the plane of motion, $\mathbf{M}_c$ are moments and α is the angular acceleration both about an axis normal to the plane of motion, I_c is the mass moment of inertia about the normal axis through the mass center, and $\boldsymbol{\rho}_{pc}$ is a vector from point p to point c.

♦ **Mass Moment of Inertia** $I = \int r^2\, dm$

Parallel-Axis Theorem $I = I_G + md^2$

Radius of Gyration $k = \sqrt{\frac{I}{m}}$

♦ **Equations of Motion**

Rigid Body	$\Sigma F_x = m(a_G)_x$
(*Plane Motion*)	$\Sigma F_y = m(a_G)_y$
	$\Sigma M_G = I_G\alpha$ or $\Sigma M_P = \Sigma(M_k)_P$

Mass Moment of Inertia

The definitions for the mass moments of inertia are

$$I_x = \int \left(y^2 + z^2\right) dm$$
$$I_y = \int \left(x^2 + z^2\right) dm$$
$$I_z = \int \left(x^2 + y^2\right) dm$$

A table listing moment of inertia formulas for some standard shapes is at the end of this section.

Parallel-Axis Theorem

The mass moments of inertia may be calculated about any axis through the application of the above definitions. However, once the moments of inertia have been determined about an axis passing through a body's mass center, it may be transformed to another parallel axis. The transformation equation is

$$I_{new} = I_c + md^2, \text{ where}$$

I_{new} = the mass moment of inertia about any specified axis

I_c = the mass moment of inertia about an axis that is parallel to the above specified axis but passes through the body's mass center

m = the mass of the body

d = the normal distance from the body's mass center to the above-specified axis

Mass Radius of Gyration

The mass radius of gyration is defined as

$$r_m = \sqrt{I/m}$$

Without loss of generality, the body may be assumed to be in the x-y plane. The scalar equations of motion may then be written as

$$\Sigma F_x = ma_{xc}$$
$$\Sigma F_y = ma_{yc}$$
$$\Sigma M_{zc} = I_{zc}\,\alpha, \text{ where}$$

zc indicates the z axis passing through the body's mass center, a_{xc} and a_{yc} are the acceleration of the body's mass center in the x and y directions, respectively, and α is the angular acceleration of the body about the z axis.

Rotation about an Arbitrary Fixed Axis

♦ **Rigid Body Motion About a Fixed Axis**

Variable α	*Constant* $\alpha = \alpha_c$
$\alpha = \frac{d\omega}{dt}$	$\omega = \omega_0 + \alpha_c t$
$\omega = \frac{d\theta}{dt}$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$
$\omega\, d\omega = \alpha\, d\theta$	$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$

For rotation about some arbitrary fixed axis q

$$\Sigma M_q = I_q\alpha$$

If the applied moment acting about the fixed axis is constant then integrating with respect to time, from $t = 0$ yields

$$\alpha = M_q/I_q$$
$$\omega = \omega_0 + \alpha t$$
$$\theta = \theta_0 + \omega_0 t + \alpha t^2/2$$

where ω_0 and θ_0 are the values of angular velocity and angular displacement at time $t = 0$, respectively.

The change in kinetic energy is the work done in accelerating the rigid body from ω_0 to ω

$$I_q\omega^2/2 = I_q\omega_0^2/2 + \int_{\theta_0}^{\theta} M_q d\theta$$

Kinetic Energy

In general the kinetic energy for a rigid body may be written as

$$T = mv^2/2 + I_c\omega^2/2$$

For motion in the xy plane this reduces to

$$T = m\left(v_{cx}^2 + v_{cy}^2\right)/2 + I_c\omega_z^2/2$$

For motion about an instant center,

$$T = I_{IC}\omega^2/2$$

♦ **Principle of Angular Impulse and Momentum**

Rigid Body	$(\mathbf{H}_G)_1 + \Sigma\int \mathbf{M}_G\, dt = (\mathbf{H}_G)_2$
(*Plane Motion*)	where $\mathbf{H}_G = I_G\omega$
	$(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\, dt = (\mathbf{H}_O)_2$
	where $\mathbf{H}_O = I_O\omega$

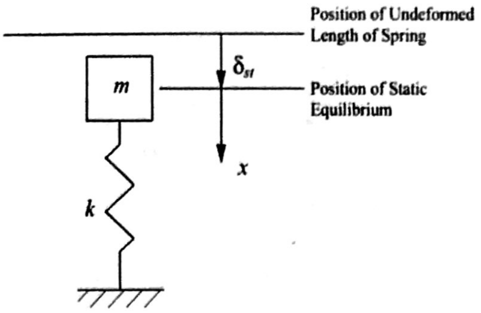
♦ **Conservation of Angular Momentum**

$\Sigma(\text{syst. }\mathbf{H})_1 = \Sigma(\text{syst. }\mathbf{H})_2$

♦ Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.

Free Vibration

The figure illustrates a single degree-of-freedom system.



The equation of motion may be expressed as

$$m\ddot{x} = mg - k(x + \delta_{st})$$

where m is mass of the system, k is the spring constant of the system, δ_{st} is the static deflection of the system, and x is the displacement of the system from static equilibrium.

From statics it may be shown that

$$mg = k\delta_{st}$$

thus the equation of motion may be written as

$$m\ddot{x} + kx = 0, \text{ or}$$

$$\ddot{x} + (k/m)x = 0$$

The solution of this differential equation is

$$x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

where $\omega_n = \sqrt{k/m}$ is the undamped natural circular frequency and C_1 and C_2 are constants of integration whose values are determined from the initial conditions.

If the initial conditions are denoted as $x(0) = x_0$ and $\dot{x}(0) = v_0$, then

$$x(t) = x_0 \cos(\omega_n t) + (v_0/\omega_n)\sin(\omega_n t)$$

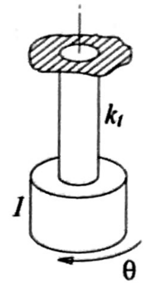
It may also be shown that the undamped natural frequency may be expressed in terms of the static deflection of the system as

$$\omega_n = \sqrt{g/\delta_{st}}$$

The undamped natural period of vibration may now be written as

$$\tau_n = 2\pi/\omega_n = \frac{2\pi}{\sqrt{\frac{k}{m}}} = \frac{2\pi}{\sqrt{\frac{g}{\delta_{st}}}}$$

Torsional Vibration



For torsional free vibrations it may be shown that the differential equation of motion is

$$\ddot{\theta} + (k_t/I)\theta = 0, \text{ where}$$

θ = the angular displacement of the system

k_t = the torsional stiffness of the massless rod

I = the mass moment of inertia of the end mass

The solution may now be written in terms of the initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$ as

$$\theta(t) = \theta_0 \cos(\omega_n t) + (\dot{\theta}_0/\omega_n)\sin(\omega_n t)$$

where the undamped natural circular frequency is given by

$$\omega_n = \sqrt{k_t/I}$$

The torsional stiffness of a solid round rod with associated polar moment-of-inertia J, length L, and shear modulus of elasticity G is given by

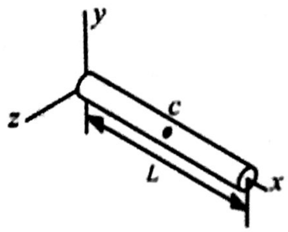
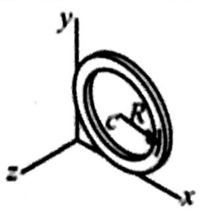
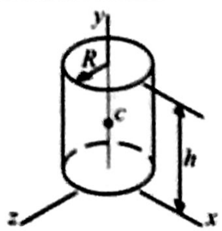
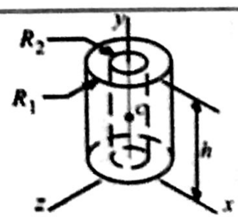
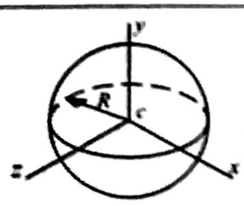
$$k_t = GJ/L$$

Thus the undamped circular natural frequency for a system with a solid round supporting rod may be written as

$$\omega_n = \sqrt{GJ/IL}$$

Similar to the linear vibration problem, the undamped natural period may be written as

$$\tau_n = 2\pi/\omega_n = \frac{2\pi}{\sqrt{\frac{k_t}{I}}} = \frac{2\pi}{\sqrt{\frac{GJ}{IL}}}$$

Figure	Mass & Centroid	Mass Moment of Inertia	(Radius of Gyration)^2	Product of Inertia
	$M = \rho LA$ $x_c = L/2$ $y_c = 0$ $z_c = 0$ A = cross-sectional area of rod ρ = mass/vol.	$I_x = I_{x_c} = 0$ $I_{y_c} = I_{z_c} = ML^2/12$ $I_y = I_z = ML^2/3$	$r_x^2 = r_{x_c}^2 = 0$ $r_{y_c}^2 = r_{z_c}^2 = L^2/12$ $r_y^2 = r_z^2 = L^2/3$	$I_{x_c y_c}$, etc. = 0 I_{xy}, etc. = 0
	$M = 2\pi R\rho A$ $x_c = R$ = mean radius $y_c = R$ = mean radius $z_c = 0$ A = cross-sectional area of ring ρ = mass/vol.	$I_{x_c} = I_{y_c} = MR^2/2$ $I_{z_c} = MR^2$ $I_x = I_y = 3MR^2/2$ $I_z = 3MR^2$	$r_{x_c}^2 = r_{y_c}^2 = R^2/2$ $r_{z_c}^2 = R^2$ $r_x^2 = r_y^2 = 3R^2/2$ $r_z^2 = 3R^2$	$I_{x_c y_c}$, etc. = 0 $I_{z_c z_c} = MR^2$ $I_{xz} = I_{yz} = 0$
	$M = \pi R^2 \rho h$ $x_c = 0$ $y_c = h/2$ $z_c = 0$ ρ = mass/vol.	$I_{x_c} = I_{z_c} = M\left(3R^2 + h^2\right)/12$ $I_{y_c} = I_y = MR^2/2$ $I_x = I_z = M\left(3R^2 + 4h^2\right)/12$	$r_{x_c}^2 = r_{z_c}^2 = \left(3R^2 + h^2\right)/12$ $r_{y_c}^2 = r_y^2 = R^2/2$ $r_x^2 = r_z^2 = \left(3R^2 + 4h^2\right)/12$	$I_{x_c y_c}$, etc. = 0 I_{xy}, etc. = 0
	$M = \pi\left(R_1^2 - R_2^2\right)\rho h$ $x_c = 0$ $y_c = h/2$ $z_c = 0$ ρ = mass/vol.	$I_{x_c} = I_{z_c}$ $= M\left(3R_1^2 + 3R_2^2 + h^2\right)/12$ $I_{y_c} = I_y = M\left(R_1^2 + R_2^2\right)/2$ $I_x = I_z$ $= M\left(3R_1^2 + 3R_2^2 + 4h^2\right)/12$	$r_{x_c}^2 = r_{z_c}^2 = \left(3R_1^2 + 3R_2^2 + h^2\right)/12$ $r_{y_c}^2 = r_y^2 = \left(R_1^2 + R_2^2\right)/2$ $r_x^2 = r_z^2$ $= \left(3R_1^2 + 3R_2^2 + 4h^2\right)/12$	$I_{x_c y_c}$, etc. = 0 I_{xy}, etc. = 0
	$M = \frac{4}{3}\pi R^3 \rho$ $x_c = 0$ $y_c = 0$ $z_c = 0$ ρ = mass/vol.	$I_{x_c} = I_x = 2MR^2/5$ $I_{y_c} = I_y = 2MR^2/5$ $I_{z_c} = I_z = 2MR^2/5$	$r_{x_c}^2 = r_x^2 = 2R^2/5$ $r_{y_c}^2 = r_y^2 = 2R^2/5$ $r_{z_c}^2 = r_z^2 = 2R^2/5$	$I_{x_c y_c}$, etc. = 0

Housner, George W., and Donald E. Hudson, *Applied Mechanics Dynamics*, D. Van Nostrand Company, Inc., Princeton, NJ, 1959. Table reprinted by permission of G.W. Housner & D.E. Hudson.