## DRAWING A TRIANGLE THAT LOOKS ACUTE AND SCALENE

As a geometry teacher, I often find myself trying to draw a "generic," or "arbitrary," triangle (i.e., one that is neither right nor isosceles). If the triangle can be obtuse, drawing it is not difficult (see **fig. 1 [Kalman]**).

If the triangle must be acute, however, it is difficult to avoid inadvertently making it look right or isosceles. For example, the triangle in **figure 2 (Kalman)** looks right, and the triangle in **figure 3 (Kalman)** looks isosceles.

For students who are fooled by appearances, such drawings can be misleading for problems or theorems requiring an arbitrary triangle. I set out to find the shape of an acute triangle that is as far from right and as far from isosceles as possible.

Since the shape of a triangle is determined by its angles, we will list three desired criteria for the angle measurements $A$, $B$, and $C$ (see **fig. 4 [Kalman]**). Assume without loss of generality that

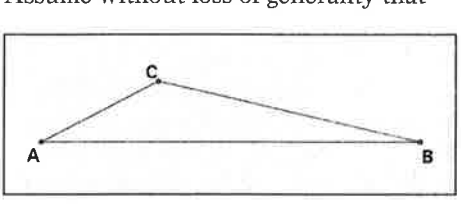


**Fig. 1 (Kalman)**

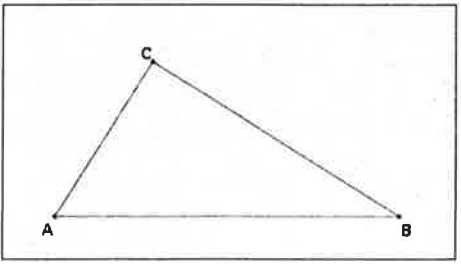


**Fig. 2** (Kalman)

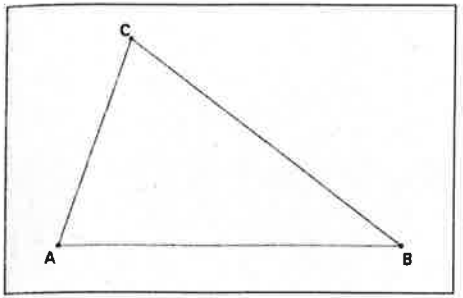


**Fig. 3 (Kalman)**

| Descriptive Criterion | Informal Implications for *A*, *B*, and *C* | Mathematical Implications for *A*, *B*, and *C* |
|---|---|---|
| 1. The triangle is valid. | $A$, $B$, and $C$ sum to 180°. | $A + B + C = 180°$. |
| 2. The triangle is acute and does not "look" right. | $A$ is much smaller than 90°. | $(90° - A)$ is as large as possible. |
| 3. The triangle does not "look" isosceles. | $B$ is not close to $A$.<br>$C$ is not close to $B$. | $\text{Min}(A - B, B - C)$ is as large as possible.* |

*Notation: $\text{Min}(x_1, x_2, \ldots, x_n)$ means the smallest of the $x_i$.

**Fig. 4 (Kalman)**

$A > B > C > 0$. Mathematically, we are seeking to maximize the value of $f(A, B, C) = \min(90° - A, A - B, B - C)$ subject to the constraint $A + B + C = 180°$.

A bit of experimentation shows that $A = 75°$, $B = 60°$, $C = 45°$ gives $f = \min(15°, 15°, 15°) = 15°$. We will argue that this solution is optimal and that it is the only optimal solution.

Assume that some solution $(A, B, C)$ has $f(A, B, C) \geq 15$.
Since $15° \leq f \leq 90° - A$, $A \leq 75°$.
Since $15° \leq f \leq A - B \leq 75° - B$, $B \leq 60°$.
Since $C = 180° - A - B \geq 180° - 75° - 60° = 45°$, $C \geq 45°$.

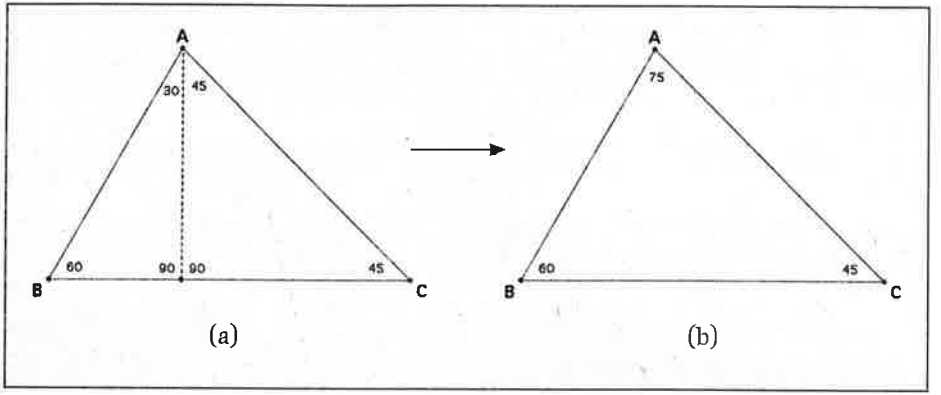


**Fig. 5 (Kalman)**

Now observe that $f \leq B - C \leq 60° - 45° = 15°$, so no solution $(A, B, C)$ can have $f(A, B, C) > 15°$. Therefore, 15° is the maximum value for $f$. Further, observe that $f = 15°$ only when equality holds in all these nonstrict inequalities—in other words, when $A = 75°$, $B = 60°$, and $C = 45°$.

How to draw this 45-60-75° triangle? It is just a 45-45-90° (isosceles right) triangle placed adjacent to a 30-60-90° triangle, both of which are familiar to a geometry teacher and relatively easy to draw (see **fig. 5 [Kalman]**). Clearly, $\triangle ABC$ is acute, and a glance will confirm that it looks nowhere near right or isosceles.

In this investigation, we chose to work directly with angle measurements, but clearly we could have listed many other criteria to define what it means to "look" arbitrary.

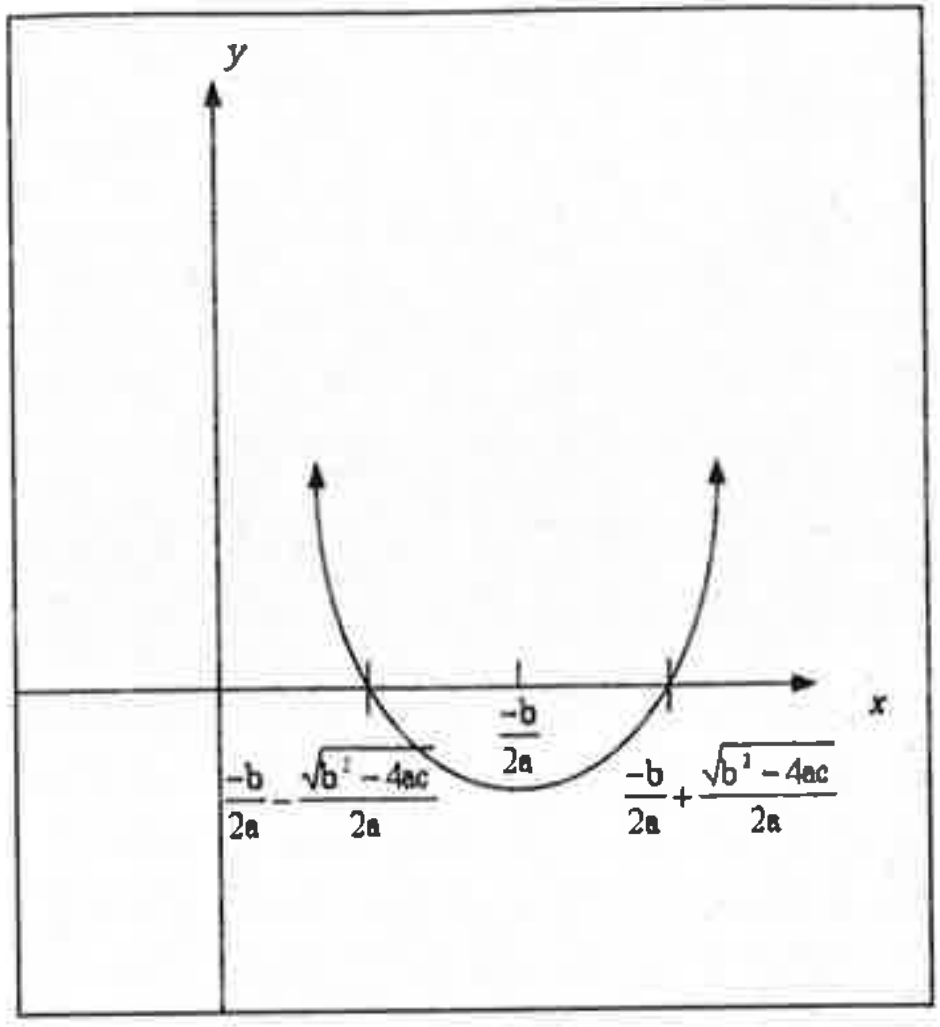


**Fig. 1 (Metz)**

Perhaps this 45-60-75° triangle will prove useful to teachers trying to draw an acute triangle that does not look special in any way.

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