Question 1

Consider the number $C(n, r)$ given by

$$C(n, r) = \frac{n!}{(n-r)! \cdot r!} ,$$

where n and r are integers satisfying $n \geq 1$ and $n \geq r \geq 0$ (recall that $0! = 1$).

(i) Suppose that $n \geq 2$ is an even integer. Find

$$\frac{C\left(n, \frac{n}{2}\right)}{C\left(n, \frac{n}{2} + 1\right)} . \qquad (1.1)$$

Your answer should be in terms of n, and in its simplest form. Justify your answer.

(ii) Using part **(i)**, justify the following statement:

$$\left| \frac{\sqrt{n}}{2^n} \cdot C\left(n, \frac{n}{2}\right) - \frac{\sqrt{n}}{2^n} \cdot C\left(n, \frac{n}{2} + 1\right) \right|$$

becomes smaller and smaller as n (being even) becomes bigger and bigger. {Note that if you use another method [not from part **(i)**], you may not be awarded full credit to this part. }

(iii) Using the Stirling formula

$$n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \qquad (n \text{ being an large integer}), \qquad (1.2)$$

and the statement that

$$\left(1 + \frac{x}{k}\right)^k \to e^x \quad \text{as} \quad k \to \infty \qquad (x \text{ being a fixed number}), \qquad (1.3)$$

find the eventual value of (that is, the limit of; in case the limit exists)

$$\frac{\sqrt{n}}{2^n} \cdot C\left(n, \frac{n}{2} + \sqrt{n}\right)$$

when the even integer $n \to \infty$ (for those integer $n \geq 4$ so that $\sqrt{n}$ is also an integer).

Your answer should be expressed in terms of π and e, and in its simplest form. Justify your answer. {Note that if you use another method [not from (1.2) or (1.3)], you may not be awarded full credit to this part. }

In (1.2) and (1.3), e is the base for the natural logarithm. You are NOT required to show (1.2) and/or (1.3).