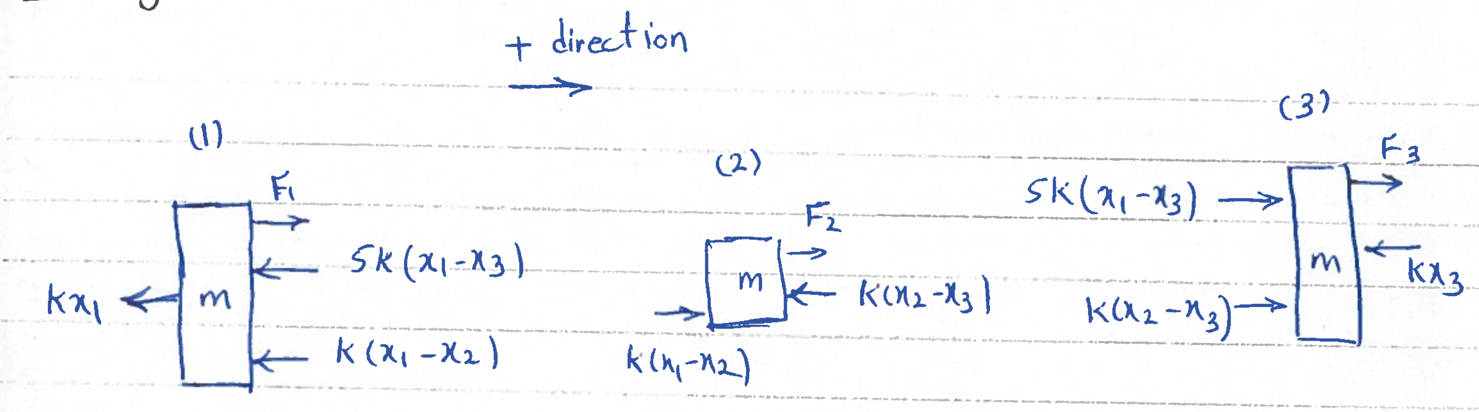
Q1 Page 1 problem set 3 solutions



$$\sum^{(1)} F = m\ddot{x}_1 \Rightarrow F_1 - kx_1 - 5k(x_1-x_3) - k(x_1-x_2) = m\ddot{x}_1$$

$$\Rightarrow m\ddot{x}_1 + kx_1 + 5k(x_1-x_3) + k(x_1-x_2) = F_1$$

$$\Rightarrow \underline{m\ddot{x}_1 + 7kx_1 - kx_2 - 5kx_3 = F_1} \quad (1)$$

$$\sum^{(2)} F = m\ddot{x}_2 \Rightarrow F_2 + k(x_1-x_2) - k(x_2-x_3) = m\ddot{x}_2$$

$$\Rightarrow \underline{m\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = F_2} \quad (2)$$

$$\sum^{(3)} F = m\ddot{x}_3 \Rightarrow F_3 + k(x_2-x_3) + 5k(x_1-x_3) - kx_3 = m\ddot{x}_3$$

$$\Rightarrow \underline{m\ddot{x}_3 - 5kx_1 - kx_2 + 7kx_3 = F_3} \quad (3)$$

Matrix Form:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 7k & -k & -5k \\ -k & 2k & -k \\ -5k & -k & 7k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Natural frequencies:

$$\left([K]-[M]\omega^2\right)\vec{X}=\vec{0}$$

$$\Rightarrow \begin{bmatrix} 7k-m\omega^2 & -k & -5k \\ -k & 2k-m\omega^2 & -k \\ -5k & -k & 7k-m\omega^2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Factoring a k and defining $\alpha=\frac{m\omega^2}{k}$, and simplifying gives:

$$\begin{bmatrix} 7-\alpha & -1 & -5 \\ -1 & 2-\alpha & -1 \\ -5 & -1 & 7-\alpha \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \left\|\begin{bmatrix} 7-\alpha & -1 & -5 \\ -1 & 2-\alpha & -1 \\ -5 & -1 & 7-\alpha \end{bmatrix}\right\|=0 \Rightarrow -a^3+16a^2-50a+24=0$$

$$\Rightarrow \alpha_1=12.0$$

$$\alpha_2=0.5858,\quad \alpha_2=3.4142$$

$$\Rightarrow \omega_1=\sqrt{12\frac{k}{m}},\quad \omega_2=\sqrt{0.5858\frac{k}{m}},\quad \omega_3=\sqrt{3.4142\frac{k}{m}}$$

Mode shapes:

$$\begin{bmatrix} 7-\alpha_1 & -1 & -5 \\ -1 & 2-\alpha_1 & -1 \\ -5 & -1 & 7-\alpha_1 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} -5 & -1 & -5 \\ -1 & -10 & -1 \\ -5 & -1 & -5 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Q1 Page 3

$$\begin{cases} -5X_1 - X_2 - 5X_3 = 0 \\ -X_1 - 10X_2 - X_3 = 0 \end{cases} \Rightarrow \begin{cases} X_2 + 5X_3 = -5X_1 \\ 10X_2 + X_3 = -X_1 \end{cases}$$

$$\Rightarrow X_2 = 0, \quad X_3 = -X_1$$

$$\Rightarrow X^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ 0 \\ -X_1 \end{Bmatrix} = X_1^{(1)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

Similarly, for α_2 and α_3:

$$X^{(2)} = X_1^{(2)} \begin{Bmatrix} 1 \\ 1.414 \\ 1 \end{Bmatrix}, \qquad X^{(3)} = X_1^{(3)} \begin{Bmatrix} 1 \\ -1.414 \\ 1 \end{Bmatrix}$$

(e): $m = 1\,kg$, $k = 100\,N/m$

(1) MATLAB: $\omega_1 = 34.6410$; $X^{(1)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$

Analytical: $\omega_1 = \sqrt{12 \times \frac{100}{1}} = 34.6410$, $X^{(1)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$

(2) MATLAB: $\omega_2 = 7.6537$, $X^{(2)} = \begin{Bmatrix} 1.0 \\ 1.4142 \\ 1.0 \end{Bmatrix}$

Analytical: $\omega_2 = \sqrt{0.5858 \times \frac{100}{1}} = 7.6538$, $X^{(2)} = \begin{Bmatrix} 1 \\ 1.414 \\ 1 \end{Bmatrix}$

(3) MATLAB: $\omega_3 = 18.4776$, $X^{(2)} = \begin{Bmatrix} 1.0 \\ -1.4142 \\ 1.0 \end{Bmatrix}$

Analytical: $\omega_3 = \sqrt{3.4142 \frac{100}{1}} = 18.4776$, $X^{(3)} = \begin{Bmatrix} 1 \\ -1.414 \\ 1 \end{Bmatrix}$

Q2 Page 1

$$\left([K]-[M]\omega^2\right)\vec{X}=\vec{0}$$

$$\begin{bmatrix} 2k-m\omega^2 & -k & 0 \\ -k & 2k-m\omega^2 & -k \\ 0 & -k & k-m\omega^2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

factoring a k and simplifing, and defining $\alpha=\frac{m\omega^2}{k}$

$$\rightarrow \begin{bmatrix} 2-\alpha & -1 & 0 \\ -1 & 2-\alpha & -1 \\ 0 & -1 & 1-\alpha \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \left|\begin{bmatrix} 2-\alpha & -1 & 0 \\ -1 & 2-\alpha & -1 \\ 0 & -1 & 1-\alpha \end{bmatrix}\right|=0 \Rightarrow -\alpha^3+5\alpha^2-6\alpha+1=0$$

$$\Rightarrow \alpha_1=0.1981,\ \alpha_2=1.5550$$
$$\alpha_3=3.2470$$

$$\Rightarrow \omega_1=\sqrt{0.1981\frac{k}{m}},\ \omega_2=\sqrt{1.5550\frac{k}{m}},\ \omega_3=\sqrt{3.2470\frac{k}{m}}$$

Mode shapes:

$$\alpha_1=0.1981 \Rightarrow \begin{bmatrix} 1.8019 & -1 & 0 \\ -1 & 1.8019 & -1 \\ 0 & -1 & 0.8019 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$1.8019X_1-X_2=0 \Rightarrow X_2=1.8019X_1$$

$$-X_2+0.8019X_3=0 \Rightarrow X_3=1.247X_2=2.2470X_1$$

$$\Rightarrow \quad X^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ 1.8019\, X_1 \\ 2.247\, X_1 \end{Bmatrix} = X_1 \begin{Bmatrix} 1 \\ 1.8019 \\ 2.247 \end{Bmatrix}$$

Similarly : $\quad X^{(2)} = \begin{Bmatrix} 1.0 \\ 0.445 \\ -0.8019 \end{Bmatrix}, \quad X^{(3)} = \begin{Bmatrix} 1.0 \\ -1.247 \\ 0.555 \end{Bmatrix}$

$$([K] - [M]\omega^2)\vec{X} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 72 - m_1\omega_1^2 & -40 & 0 \\ -40 & 120 - m_2\omega_1^2 & -80 \\ 0 & -80 & 144 - m_3\omega_1^2 \end{bmatrix} \vec{X}^{(1)} = \vec{0}$$

given:

$$\omega_1 = 3.6703, \quad X^{(1)} = \begin{Bmatrix} 1.0 \\ 1.1265 \\ 1.0 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 72 - 13.4711\, m_1 & -40 & 0 \\ -40 & 120 - 13.4711\, m_2 & -80 \\ 0 & -80 & 144 - 13.4711\, m_3 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 1.1265 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow 72 - 13.4711\, m_1 - 40 \times 1.1265 = 0 \Rightarrow m_1 = 2.0$$

$$-40 + 1.1265 \times (120 - 13.4711\, m_2) - 80 = 0 \Rightarrow m_2 = 1.0$$

$$-80 \times 1.1265 + 144 - 13.4711\, m_3 = 0 \Rightarrow m_3 = 4.0$$

$$\Rightarrow [M] = \begin{bmatrix} 2.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 4.0 \end{bmatrix}$$

Other natural frequencies:

$$\left| \begin{bmatrix} 72 - 2\omega^2 & -40 & 0 \\ -40 & 120 - \omega^2 & -80 \\ 0 & -80 & 144 - 4\omega^2 \end{bmatrix} \right| = 0$$

defining $\lambda = \omega^2 \Rightarrow -8\lambda^3 + 1536\lambda^2 - 60288\lambda + 552960 = 0$

$$\Rightarrow \begin{cases} \lambda_1 = 13.4709 \Rightarrow \omega_1 = 3.6703 \rightarrow \text{already given} \\ \lambda_2 = 36 \Rightarrow \omega_2 = 6 \\ \lambda_3 = 142.5291 \Rightarrow \omega_3 = 11.9386 \end{cases}$$

Mode shapes:

$$X^{(2)} \Rightarrow \begin{bmatrix} 72-2\omega_2^2 & -40 & 0 \\ -40 & 120-\omega_2^2 & -80 \\ 0 & -80 & 144-4\omega_2^2 \end{bmatrix} X^{(2)} = \vec{0}$$

$$\rightarrow \begin{bmatrix} 0 & -40 & 0 \\ -40 & 84 & -80 \\ 0 & -80 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow -40 X_2 = 0 \Rightarrow X_2 = 0$$

$$-40 X_1 + \underbrace{84 X_2}_{=0} - 80 X_3 = 0 \Rightarrow X_3 = -0.5 X_1$$

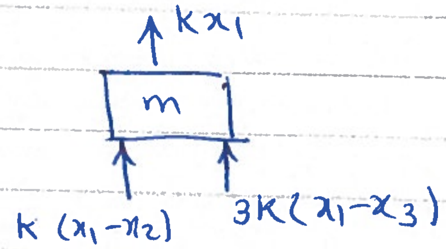
$$\Rightarrow X^{(2)} = \begin{Bmatrix} X_1 \\ 0 \\ -0.5 X_1 \end{Bmatrix} = X_1 \begin{Bmatrix} 1 \\ 0 \\ -0.5 \end{Bmatrix}$$

$\Rightarrow$ Similarly :

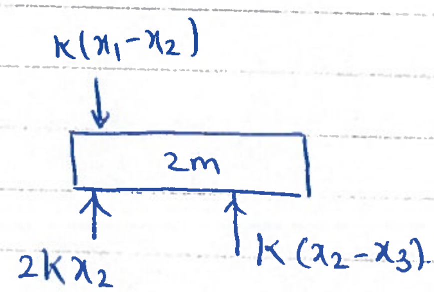
$$X^{(3)} = \begin{Bmatrix} 1.0 \\ -5.3265 \\ 1.0 \end{Bmatrix}$$

Q4 - Page 1

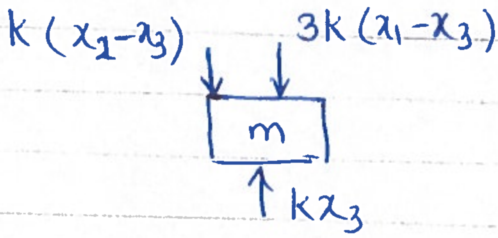
Free-body-diagram: $\downarrow$ + direction



(1)



(2)



(3)

Equations of motion:

(1): $\sum^{(1)} F = m\ddot{x}_1 \Rightarrow -kx_1 - k(x_1-x_2) - 3k(x_1-x_3) = m\ddot{x}_1$

$\Rightarrow m\ddot{x}_1 + kx_1 + k(x_1-x_2) + 3k(x_1-x_3) = 0$

$\Rightarrow m\ddot{x}_1 + 5kx_1 - kx_2 - 3kx_3 = 0$

(2) $\sum^{(2)} F = 2m\ddot{x}_2 \Rightarrow k(x_1-x_2) - 2kx_2 - k(x_2-x_3) = 2m\ddot{x}_2$

$\Rightarrow 2m\ddot{x}_2 - kx_1 + 4kx_2 - kx_3 = 0$

(3) $\sum^{(3)} F = m\ddot{x}_3 \Rightarrow k(x_2-x_3) + 3k(x_1-x_3) - kx_3 = m\ddot{x}_3$

$\Rightarrow m\ddot{x}_3 - 3kx_1 - kx_2 + 5kx_3 = 0$

Q4 - Page 2

Matrix form:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 5k & -k & -3k \\ -k & 4k & -k \\ -3k & -k & 5k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Natural frequencies:

$$([K] - [M]\omega^2)\vec{X} = \vec{c}$$

$$\begin{bmatrix} 5k - m\omega^2 & -k & -3k \\ -k & 4k - 2m\omega^2 & -k \\ -3k & -k & 5k - m\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

factoring a k, defining $\alpha = \frac{m\omega^2}{k}$, and simplifying:

$$\begin{bmatrix} 5 - \alpha & -1 & -3 \\ -1 & 4 - 2\alpha & -1 \\ -3 & -1 & 5 - \alpha \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \left| \begin{bmatrix} 5-\alpha & -1 & -3 \\ -1 & 4-2\alpha & -1 \\ -3 & -1 & 5-\alpha \end{bmatrix} \right| = 0 \Rightarrow -2\alpha^3 + 24\alpha^2 - 70\alpha + 48 = 0$$

$$\Rightarrow \begin{cases} \alpha_1 = 1 & \rightarrow \quad \omega_1 = \sqrt{\frac{k}{m}} \\ \alpha_2 = 3 & \rightarrow \quad \omega_2 = \sqrt{3\frac{k}{m}} \\ \alpha_3 = 8 & \rightarrow \quad \omega_3 = \sqrt{8\frac{k}{m}} \end{cases}$$

Q4 - Page 3

Mode shapes:

$$\alpha_1 = 1 \rightarrow \begin{bmatrix} 5-\alpha_1 & -1 & -3 \\ -1 & 4-2\alpha_1 & -1 \\ -3 & -1 & 5-\alpha_1 \end{bmatrix} X^{(1)} = \vec{0}$$

$$\rightarrow \begin{bmatrix} 4 & -1 & -3 \\ -1 & 2 & -1 \\ -3 & -1 & 4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} 4X_1 - X_2 - 3X_3 = 0 \\ -X_1 + 2X_2 - X_3 = 0 \end{cases} \rightarrow \begin{cases} X_2 + 3X_3 = 4X_1 \\ 2X_2 - X_3 = X_1 \end{cases} \Rightarrow \begin{matrix} X_2 = X_1 \\ X_3 = X_1 \end{matrix}$$

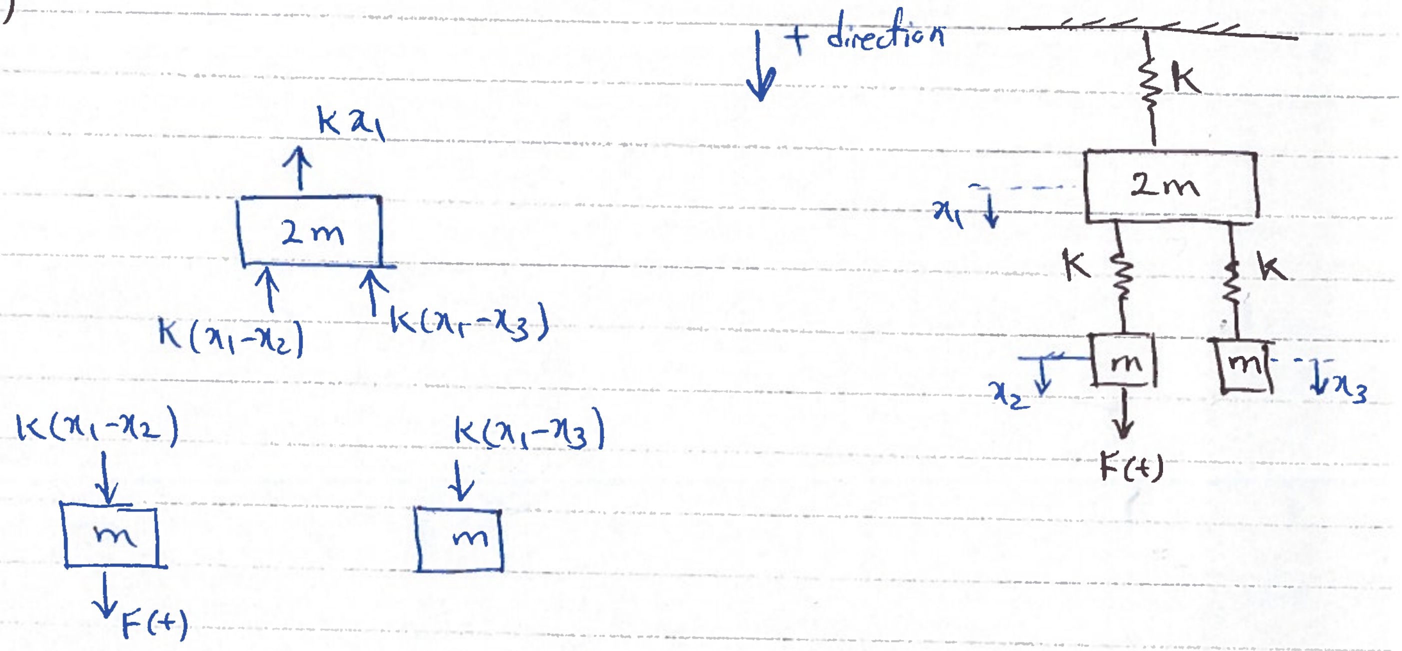
$$\Rightarrow X^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_1 \\ X_1 \end{Bmatrix} = X_1 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Similarly: $\alpha_2 = 3 \Rightarrow \dots \Rightarrow X^{(2)} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$

$$\alpha_3 = 8 \Rightarrow \dots \Rightarrow X^{(3)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

Q5 - Page 1

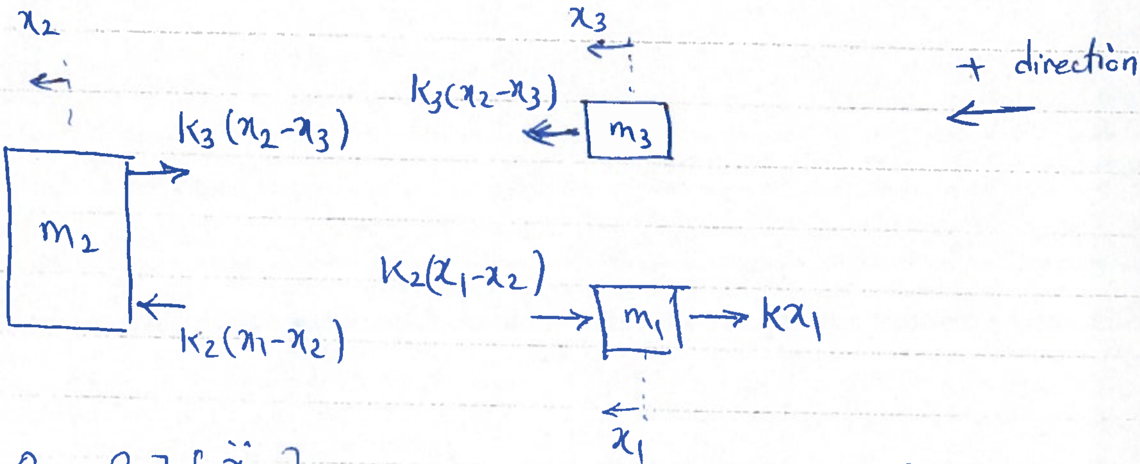
(a)



Matrix form of equations of motion (not asked in the question - For practice)

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 3K & -K & -K \\ -K & K & 0 \\ -K & 0 & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}$$

(b)



$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 & 0 \\ -K_2 & K_2+K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$