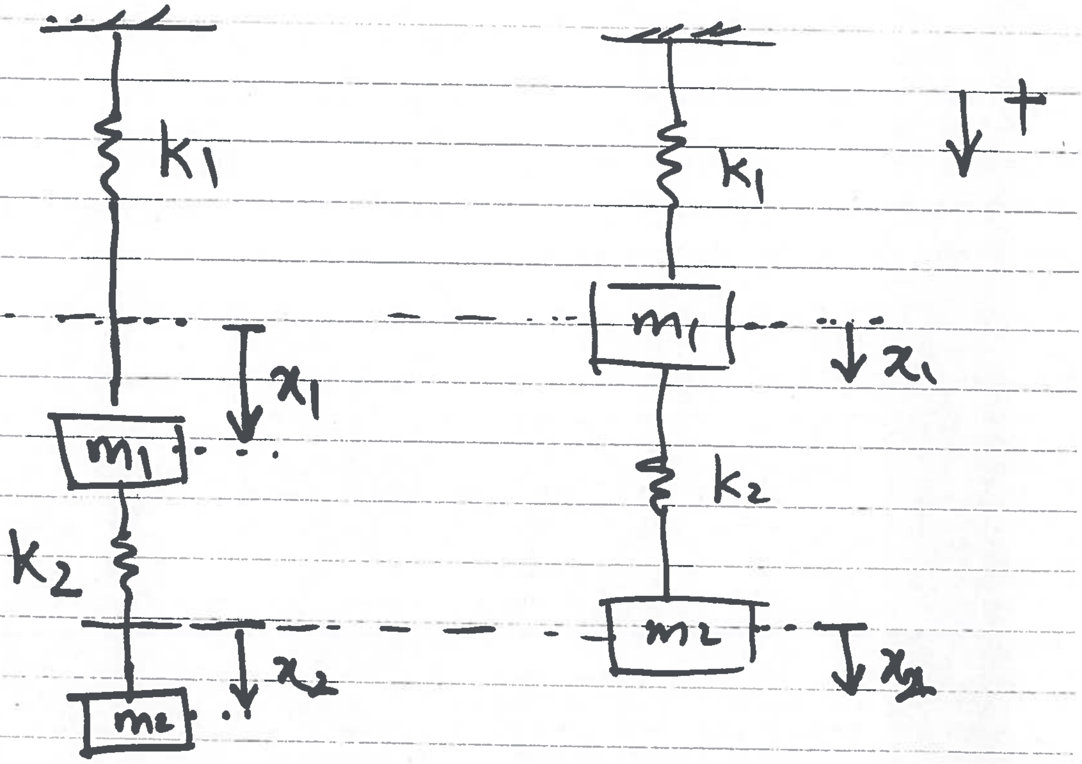
Q1 — Page 1

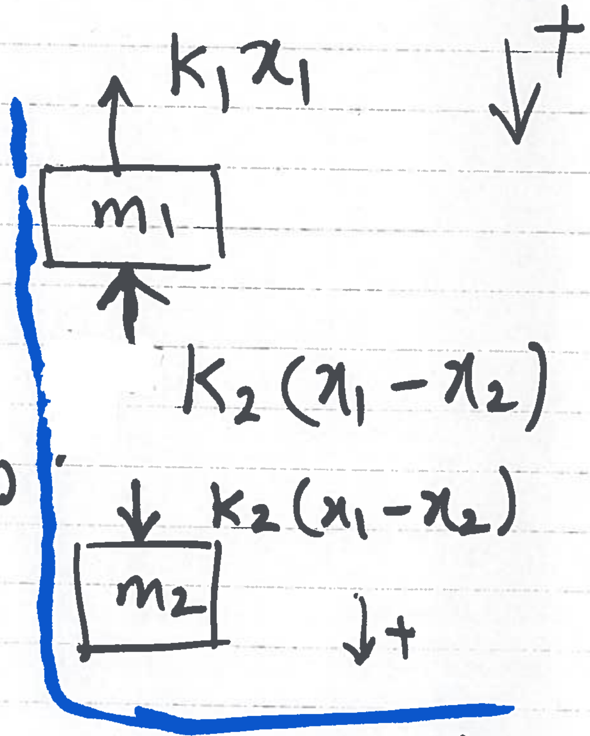


mass 1

$$\sum F_x = m_1 \ddot{x}_1$$

$$-k_1 x_1 - k_2 (x_1 - x_2) = m_1 \ddot{x}_1$$

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = 0$$



$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \quad (1)$$

$\max 2$

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$$\sum F_x = m_2 \ddot{x}_2$$

$$k_2(x_1 - x_2) = m_2 \ddot{x}_2$$

$$\Rightarrow m_2\ddot{x}_2 - k_2(x_1 - x_2) = 0$$

$$\underline{m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0} \quad (2)$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$m_1 = m$, $m_2 = 2m$

$k_1 = k$, $k_2 = 2k$

$$[M] = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}, \quad [K] = \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix}$$

$$\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = X_1 \cos(\omega t + \phi) \\ x_2 = X_2 \cos(\omega t + \phi) \end{cases}$$

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \Rightarrow [M]\ddot{\vec{x}} + [K]\vec{x} = \vec{0}$$

$$\begin{cases} \ddot{x}_1 = -\omega^2 X_1 \cos(\omega t + \phi) = -\omega^2 x_1 \\ \ddot{x}_2 = -\omega^2 X_2 \cos(\omega t + \phi) = -\omega^2 x_2 \end{cases}$$

$$\Rightarrow \underline{\ddot{\vec{x}} = -\omega^2 \vec{x}} \qquad \dot{\square} = \frac{d}{dt}$$

$$\cancel{x = [M]\omega^2 + [K]}$$

$$-[M]\omega^2 \vec{x} + [K]\vec{x} = \vec{0}$$

$$\underline{([K] - [M]\omega^2)\vec{x} = \vec{0}} \quad \#$$

$$\# \quad A\vec{x} = \vec{0} \Rightarrow \begin{cases} \text{trivial sol.} \Rightarrow \vec{x} = \vec{0} \\ \text{non-trivial sol} \rightarrow |A| = 0 \end{cases}$$

$$[M]\omega^2 = \begin{bmatrix} m\omega^2 & 0 \\ 0 & 2m\omega^2 \end{bmatrix}$$

$$[K] - [M]\omega^2$$

$$= \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix} - \begin{bmatrix} m\omega^2 & 0 \\ 0 & 2m\omega^2 \end{bmatrix}$$

$$\Rightarrow \left| \begin{bmatrix} 3k - m\omega^2 & -2k \\ -2k & 2k - 2m\omega^2 \end{bmatrix} \right| = 0$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow |A| = ad - bc$$

$$\left| k \begin{bmatrix} 3 - \frac{m\omega^2}{k} & -2 \\ -2 & 2 - 2\frac{m\omega^2}{k} \end{bmatrix} \right| = 0$$

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$$\left| \begin{bmatrix} 3-\alpha & -2 \\ -2 & 2-2\alpha \end{bmatrix} \right| = 0$$

$$\alpha = \frac{m\omega^2}{k}$$

$$(3-\alpha)(2-2\alpha) - (-2)(-2) = 0$$

$$6 - 6\alpha - 2\alpha + 2\alpha^2 - 4 = 0$$

$$2\alpha^2 - 8\alpha + 2 = 0$$

$$\Rightarrow \alpha_1 = 3.73$$

$$\alpha_2 = 0.27$$

$$\alpha_1 = \frac{m\omega_1^2}{k} = 3.73 \Rightarrow \omega_1^2 = 3.73\frac{k}{m}$$

$$\Rightarrow \omega_1 = \sqrt{3.73\frac{k}{m}}$$

$$\alpha_2 = \frac{m\omega_2^2}{k} = 0.27$$

$$\Rightarrow \omega_2 = \sqrt{0.27\frac{k}{m}}$$

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$$\begin{bmatrix} 3k - m\omega^2 & -2k \\ -2k & 2k - 2m\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 3k - \cancel{m}\dfrac{3.73\,k}{\cancel{m}} & -2k \\ -2k & 2k - 2\cancel{m}\dfrac{3.73\,k}{\cancel{m}} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$\begin{bmatrix} -0.73\,k & -2k \\ -2k & -5.46k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$-0.73\,X_1 - 2X_2 = 0$$

$$X_2 = -0.365\,X_1$$

$$-2X_1 - 5.46\,X_2 = 0$$

$$\Rightarrow X_2 = -0.366\,X_1$$

$$X^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ -0.366\,X_1 \end{Bmatrix} = X_1 \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix}$$

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Second Mode shape can be

obtained as: $X^{(2)} = X_1 \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix}$

Q2 Page 1

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad [K] = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\omega_1 = \sqrt{2}, \quad X^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$m_1, m_2, \omega_2, X^{(2)} = ?$$

$$[M]\omega^2 = \begin{bmatrix} m_1\omega^2 & 0 \\ 0 & m_2\omega^2 \end{bmatrix}$$

$$[K] - [M]\omega^2 = \Big[\ \cancel{27} \ -$$

$$\begin{bmatrix} 27 - m_1\omega^2 & -3 \\ -3 & 3 - m_2\omega^2 \end{bmatrix} \vec{X} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 27 - 2m_1 & -3 \\ -3 & 3 - 2m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Q2 Page 2

$$27 - 2m_1 + 3 = 0 \rightarrow 30 - 2m_1 = 0$$
$$\rightarrow m_1 = 15$$

$$-3 - 3 + 2m_2 = 0 \rightarrow m_2 = 3$$

$$\begin{bmatrix} 27 - 15\omega^2 & -3 \\ -3 & 3 - 3\omega^2 \end{bmatrix} \vec{X} = \vec{0}$$

$$\left| \begin{bmatrix} 27 - 15\lambda & -3 \\ -3 & 3 - 3\lambda \end{bmatrix} \right| = 0$$

$$\omega^2 = \lambda$$

$$(27 - 15\lambda)(3 - 3\lambda) - (-3)(-3) = 0$$

$$\rightarrow \lambda_1 = 2 = \omega_1^2 \rightarrow \omega_1 = \sqrt{2}$$

$$\lambda_2 = 0.8 = \omega_2^2 \Rightarrow \omega_2 = \sqrt{0.8}$$

$$X^{(2)} = \begin{Bmatrix} 1 \\ 5 \end{Bmatrix}$$

Q3 Page 1

$$[M] = \begin{bmatrix} m & 0 \\ 0 & 4 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k & -4 \\ -4 & 12 \end{bmatrix}$$

relation between k and m

such that : $X = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

$$\left([K] - [M]\omega^2\right) \vec{X} = \vec{0}$$

$$\left(\begin{bmatrix} k & -4 \\ -4 & 12 \end{bmatrix} - \begin{bmatrix} m & 0 \\ 0 & 4 \end{bmatrix}\omega^2\right) \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k - m\omega^2 & -4 \\ -4 & 12 - 4\omega^2 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Q3 Page 2

$$K - m\omega^2 + 4 = 0$$

$$-4 - 12 + 4\omega^2 = 0 \Rightarrow \omega^2 = 4$$
$$\omega = 2$$

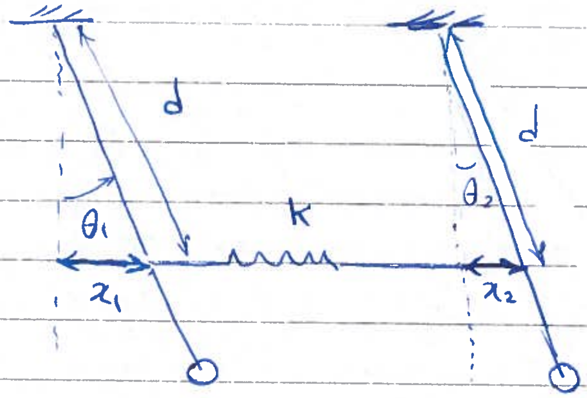
$$k - 4m + 4 = 0$$

$$\rightarrow \boxed{k = 4m - 4}$$

$$m = \frac{k}{4} + 1$$

Q 4 – page 1

Spring is compressed by magnitude of $x_1 - x_2$ assuming $x_1 > x_2$



Assuming small angles:

$\sin\theta_1 = \theta_1$, $\sin\theta_2 = \theta_2$

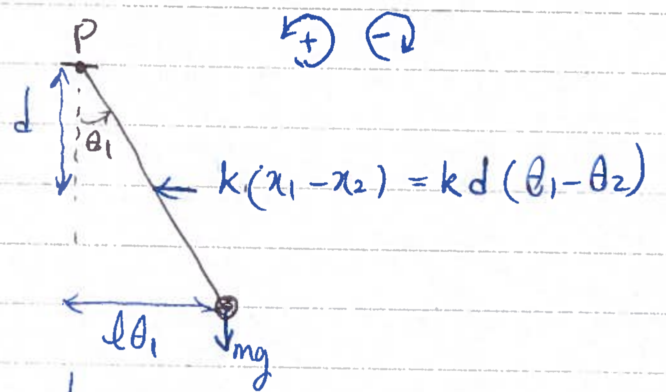
$\cos\theta_1 = 1$, $\cos\theta_2 = 1$

$\sin\theta_1 = \frac{x_1}{d}$

$\Rightarrow \theta_1 = \frac{x_1}{d} \rightarrow \boxed{x_1 = d\theta}$

Similarly : $\boxed{x_2 = d\theta_2}$

Free-body-diagram:

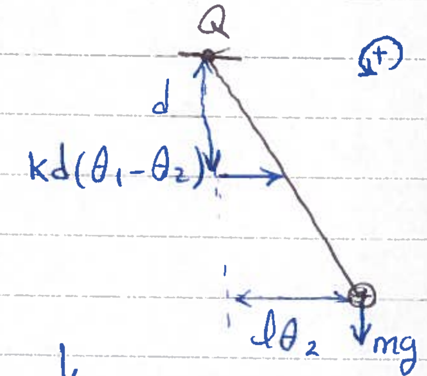


$\sum M_p = I_p \ddot{\theta}_1$, $I_p = ml^2$

$\Rightarrow -kd(\theta_1 - \theta_2)d - mgl\theta_1 = ml^2\ddot{\theta}_1$

$\Rightarrow ml^2\ddot{\theta}_1 + mgl\theta_1 + kd^2(\theta_1 - \theta_2) = 0$

$\Rightarrow ml^2\ddot{\theta}_1 + (mgl + kd^2)\theta_1 - kd^2\theta_2 = 0$ (1)



$\sum M_Q = I_Q \ddot{\theta}_2$, $I_Q = ml^2$

$kd(\theta_1 - \theta_2)d - mgl\theta_2 = ml^2\ddot{\theta}_2$

$\Rightarrow ml^2\ddot{\theta}_2 + mgl\theta_2 - kd^2(\theta_1 - \theta_2) = 0$

$\Rightarrow ml^2\ddot{\theta}_2 + (mgl + kd^2)\theta_2 - kd^2\theta_1 = 0$ (2)

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Equations of motion:

$$\begin{cases} ml^2\ddot{\theta}_1 + (mgl + kd^2)\theta_1 - kd^2\theta_2 = 0 \\ ml^2\ddot{\theta}_2 - kd^2\theta_1 + (mgl + kd^2)\theta_2 = 0 \end{cases}$$

$$\Rightarrow \underbrace{\begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 \end{bmatrix}}_{[M]} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} mgl + kd^2 & -kd^2 \\ -kd^2 & mgl + kd^2 \end{bmatrix}}_{[K]} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\vec{\theta} = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

$$\Rightarrow ([K] - [M]\omega^2)\vec{\theta} = \vec{0} \Rightarrow \left| [K] - [M]\omega^2 \right| = 0$$

$$\Rightarrow \Rightarrow \left| \begin{bmatrix} mgl + kd^2 - ml^2\omega^2 & -kd^2 \\ -kd^2 & mgl + kd^2 - ml^2\omega^2 \end{bmatrix} \right| = 0$$

Factor mgl: $\Rightarrow \left| \begin{bmatrix} 1 + \frac{kd^2}{mgl} - \frac{l}{g}\omega^2 & -\frac{kd^2}{mgl} \\ -\frac{kd^2}{mgl} & 1 + \frac{kd^2}{mgl} - \frac{l}{g}\omega^2 \end{bmatrix} \right| = 0$

define $\frac{l}{g}\omega^2 = \alpha$

$\frac{kd^2}{mgl} = C$

$$\Rightarrow \left| \begin{bmatrix} 1 + C - \alpha & -C \\ -C & 1 + C - \alpha \end{bmatrix} \right| = 0$$

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$$(1+C-\alpha)^2 - C^2 = 0$$

$$\Rightarrow \alpha^2 - (2+2C)\alpha + 2C+1 = 0$$

$$\alpha_1 = 1$$

$$\alpha_2 = 2C+1$$

$$\Rightarrow \alpha_1 = \frac{l}{g}\omega_1^2 = 1 \rightarrow \omega_1^2 = \frac{g}{l} \rightarrow \omega_1 = \sqrt{g/l}$$

$$\alpha_2 = \frac{l}{g}\omega_2^2 = \frac{2kd^2}{mgl} + 1 \rightarrow \omega_2^2 = \frac{g}{l} + \frac{2kd^2}{ml^2}$$

$$\rightarrow \omega_2 = \sqrt{\frac{g}{l} + \frac{2kd^2}{ml^2}}$$

Substitute the two ω values in $\left([K] - [M]\omega^2\right)\vec{\theta} = \vec{0}$

and you will obtain mode shapes of the system as:

$$\omega_1 = \sqrt{g/l} \rightarrow \vec{\theta}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2kd^2}{ml^2}} \Rightarrow \theta^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$