Problem 4. (3 points) A spaceship in three-dimensional space is equipped with a set of thrusters, described by a configuration matrix T, whose column vectors $\vec{t}_i$ give the orientation (as vector direction) and power (as vector magnitude) of each thruster. Thus,

$$T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 3 & 5 & 1 \end{pmatrix}$$

describes a configuration with three thrusters. If the ship is positioned at $(0,0,0)$, the first thruster is pointing towards coordinates $(1,2,3)$ and has power $\left|\vec{t}_1\right| = \sqrt{14}$.

The thrusters are operated with a command vector $\vec{c} = \langle c_1, c_2, c_3 \rangle^T$, where c_i describes the activation of thruster i. For example, $\vec{c} = \langle 1, -2, 0 \rangle^T$ activates thruster 1 with power 1, thruster 2 with twice that power but in the "reverse thrust" mode, and leaves thruster 3 unactivated.

(All vectors in this problem are column vectors, and we will be omitting the transpose symbol from now on.)

The product $T\vec{c}$ gives a displacement vector $\vec{d}$, describing the change in the ship's coordinates after a single firing of all thrusters. In our example, $\vec{d} = \langle -3, -4, -7 \rangle$. If the ship was positioned at $(0,0,0)$, it will now be at $(-3,-4,-7)$. Issuing a subsequent command $\vec{c} = \langle 1,1,1 \rangle$ will give $\vec{d} = \langle 4,5,9 \rangle$ and bring the ship to $(1,1,2)$.

Use Python, NumPy, and SciPy to write the following functions. When you need to provide specific outputs, use the T defined above. At the same time, your functions need to be generic and work with any T.

(a) Write function `distance`$(T, \vec{c})$ that returns the distance traveled by the ship after a single firing of all thrusters.

Determine the distances resulting from commands $\langle 1,1,1 \rangle$, $\langle -1,-1,-1 \rangle$, $\langle 1,2,3 \rangle$, $\langle 3,2,1 \rangle$.

(b) Write function `reachable`(T, p) that returns `True` if the ship starting at $(0,0,0)$ can reach point p in any number of moves, or `False` otherwise.

Determine whether the following points are reachable: $(0,2,2)$, $(1,1,-1)$, $(-5,1,-4)$, $(1,3,1)$.

(c) Commands that result in the displacement vector $\vec{d} = \langle 0,0,0 \rangle$ stress the ship's structure and are considered dangerous. Write function `safe`$(T, \vec{c})$ that returns `True` if the command is safe, or `False` otherwise.

Note that computing $\vec{d} = T\vec{c}$ directly means activating the thrusters with a potentially dangerous command and is not acceptable, so we need to find another way.

Determine the safety of commands $\langle 4,-7,3 \rangle$, $\langle 0,0,0 \rangle$, $\langle 0,2,5 \rangle$, $\langle 3,-2,1 \rangle$.

(d) Write function `equivalent`(T_1, T_2) that returns `True` if configurations T_1 and T_2 are equivalent (that is, can generate the same displacement vectors), or `False` otherwise.

Determine which of the following configurations are equivalent to T:

$$\begin{pmatrix} 3 & 3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 & 7 \\ 5 & 1 & 2 \\ 7 & 4 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 1 \end{pmatrix}.$$