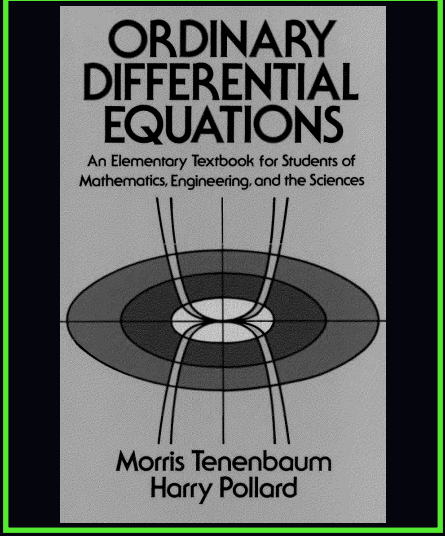
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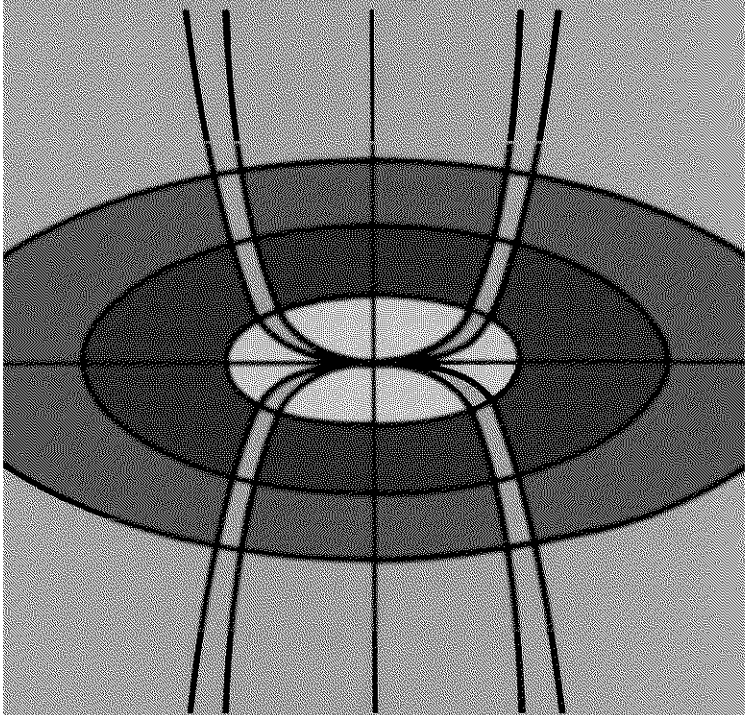
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ORDINARY DIFFERENTIAL EQUATIONS

An Elementary Textbook for Students of Mathematics, Engineering, and the Science



Morris Tenenbaum
Harry Pollard

ORDINARY DIFFERENTIAL EQUATIONS

An Elementary Textbook for Students of Mathematics, Engineering, and the Sciences

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Cornell University

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DOVER PUBLICATIONS, INC., NEW YORK

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This Dover edition, first published in 1985, is an unabridged and corrected republication of the work first published by Harper & Row, Publishers, Inc., New York, in 1963.

Manufactured in the United States of America
Dover Publications, Inc., 31 East 2nd Street, Mineola, N.Y. 11501

Library of Congress Cataloging in Publication Data

Tenenbaum, Morris.
Ordinary differential equations.

Reprint. Originally published: New York : Harper & Row, 1963.
Bibliography: p.
Includes index.
1. Differential equations. I. Pollard, Harry, 1919– II. Title.
QA372.T4 1985 515.3'5 85-12983
ISBN 0-486-64940-7

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