NUMBER DEFINITIONS

By **CHARLES BRUMFIEL**

University of Michigan
Ann Arbor, Michigan

AFTER reading David Rappaport's article in the March 1970 MATHEMATICS TEACHER, I turned to my dictionary and found the following definitions:

whole number . . . an integer
integer . . . a whole number

Under *integer* I was referred to *complex integer*, and this definition was:

complex integer . . . a number $a + bi$ where a and b are real integers.

But real integers are not defined in my dictionary!

As badly as we handle our definitions, dictionaries do worse. This is some consolation.

I feel somewhat involved in the *whole number–counting number* terminology. University mathematicians have for centuries used *whole number* and *integer* interchangeably. I am sure that most still do so. Hence, when we began to write school texts in 1955 on the Ball State Project, I did not feel free to refer to the set of cardinal numbers, {0, 1, 2, . . .}, as the set of whole numbers. I wish now I had used the simple terminology *cardinal numbers*, but this seemed pretentious and I used *counting numbers* instead. A few years later, SMSG decreed that *whole numbers* should be used. I suggest that the phrase *counting numbers* be gracefully withdrawn from circulation.

It does seem desirable to have a short name for the positive integers, and *natural numbers* is as good as any. We should understand, however, that mathematicians have carried along two definitions for centuries. Some include zero in the set of natural numbers and some do not. This condition will persist. It is best to tell students the truth. One of the worst things you can do in teaching mathematics is to treat definitions as if they were "right" or "wrong."

I think I see the wave of the future. We shall tell children about integers in kindergarten. Having done so, we shall be able to speak of the *positive integers* and the *negative integers* from the beginning. I think this will be ill advised.

A short time ago I visited a junior high school class. A student argued that -3 *is* a whole number because it *is not broken*. The teacher could not satisfy him. I had the morbid desire to tell the student that -3 is still a whole number to most university mathematicians, excluding those who wrote for SMSG or who have been trained in the new mathematics, but I refrained. Actually, I feel that it is healthy to discuss definitions, and all teachers, including elementary teachers, should utilize opportunities that arise in the classroom to do so. The task of mathematics educators is to train teachers so that they can manage such discussions.

Mr. Rappaport devoted most of his energies to fraction and rational-number terminology. I share his regret that this language is so messy. I don't know what

*From *The Mathematics Teacher*, April 1972, pp. 313-314.

to do about it. We should teach as many teachers as possible the standard mathematical set-theoretic definitions of ordinal numbers, cardinal numbers, and rational numbers. Every teacher who understands these definitions will realize that they are not meant for children. We must tell children something else.

One basic difficulty in choosing a school terminology for fractions and rational numbers is that so many writers wish to pretend to give precise definitions. Let's face it. The mathematical definitions are too complicated. Let's concentrate on making teachers and children comfortable without lying to them.

Why not let the terminology flow along with the use of the symbols. When the emphasis is on the use of fractions to describe the real world—and this is surely the major emphasis in the six elementary grades—just let a fraction be a symbol, written by separating two whole-number symbols by a bar and used to compare two "things" in the real world. The fraction $\frac{2}{3}$ may compare a set of boys to a set of girls, or a line segment to a unit segment, or a rectangular region to a unit region, and so on. This is the only meaning one needs to ascribe to a fraction in its use to describe the real world.

Late in the elementary school years, and in junior high school, teachers can note that for one physical situation it may be natural to use any one of many fractions to compare two given sets. The concept of *equivalent fractions* is a natural one, and we can agree that a rational number is "associated" with each equivalence class. There is no need to insist that the rational number *is* the equivalence class. This is the standard mathematical definition; but when this set-theoretic definition is formulated in a careful mathematical development, it has been presaged by a sequence of meticulous definitions, including the concepts of *ordinal number* and *ordered pair*. There is something phony about pretending to present precise definitions at this level. However, I have no objection if teachers prefer to say that the rational number *is* the equivalence class of fractions, even though the concept of an *equivalence class of symbols* is a bit hazy.

The great number-numeral controversy seems to be running its course. I shall be glad when it has faded away. Through the elementary school years, numbers are to most children simply symbols that describe the real world. This is the way we should talk about numbers. The child unblushingly adds and multiplies symbols. Don't give him a guilt feeling by telling him that he can *write* symbols but he can't *add* them, that he can *add* numbers but he can't *write* them. If you really think carefully about the process by which the game of arithmetic is abstracted from the real world, you may come to the conclusion that there is a stage when addition and multiplication really are things we do with symbols.

I do wish that someone would take his tape recorder into the heartland of America and record the definitions that children and teachers would formulate if they were asked to define whole number, natural number, integer, fraction, rational number, and so on. Mathematics educators might be greatly surprised if they knew what children and teachers really think about these things.