Oct 1

Cross product.

$\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

Find $\vec{c}$ s.t $\vec{a} \cdot \vec{c} = 0$, $\vec{b} \cdot \vec{c} = 0$

$\vec{c} = \langle c_1, c_2, c_3 \rangle$

$\Rightarrow$ $a_1c_1 + a_2c_2 + a_3c_3 = 0$

$b_1c_1 + b_2c_2 + b_3c_3 = 0$.

$\vec{c} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$ is a solution

Define $\vec{a} \times \vec{b} = \vec{c} = \langle \cdots \rangle$

- determinant of order 2. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{\text{definition}}{=} ad - bc$

- ... order 3. definition.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$= \cdots$.

Take $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = \cdots$

rewrite $\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ is a vector

Ex. $\vec{a} = \langle 2, 7, -5 \rangle$, $\vec{b} = \langle 1, 3, 4 \rangle$

$\Rightarrow \vec{a} \times \vec{b} = \langle 43, -13, -1 \rangle$

Ex. $\vec{a} \times \vec{a} = \vec{0}$.

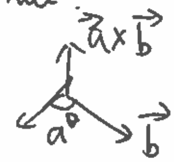
Thm $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$

Right

Geometric meaning.

- right hand rule.



- length. θ angle between $\vec{a}, \vec{b}$

$0 \le \theta \le \pi$

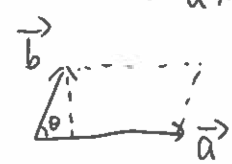
$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin\theta$.

- Proof Expand $|\vec{a} \times \vec{b}|^2$.

use The cosine Law.

- If $\vec{a}$ parallel to $\vec{b}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{0}$

- 

area $= |\vec{a} \times \vec{b}|$

- $\vec{i} \times \vec{j} = \vec{k}$, ...

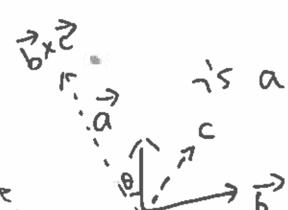
$\vec{j} \times \vec{i} = -\vec{k}$.

- Properties P859 $\quad \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

- Triple product.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is a number



of parallelepiped volume

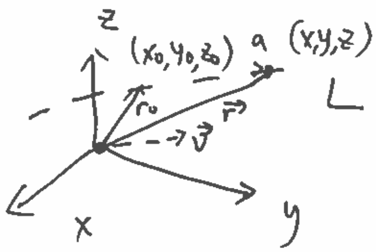
- $\vec{a}, \vec{b}, \vec{c}$ are coplanar if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

$\vec{b} \times \vec{c}$ = area $\Rightarrow$ volume

$|\vec{a}| \cos\theta$ = height

Oct 3

Equation of Lines



$$\begin{cases} \vec{a} + \vec{r_0} = \vec{r} \\ \vec{a} = t\vec{v} \end{cases} \qquad \vec{v} = \langle a,b,c \rangle$$

$\Rightarrow \vec{r} = \vec{r_0} + t\vec{v}$ is a vector eq of L.

↑ parameter

parallelepiped

Volume

$|\vec{a}\cdot(\vec{b}\times\vec{c})|$ is the volume

$$\Rightarrow \langle x,y,z \rangle = \langle x_0+ta,\ y_0+tb,\ z_0+tc \rangle$$

Parametric equations of a line through (x_0,y_0,z_0) and // $\langle a,b,c \rangle$ are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

a, b, c are direction #s.

E.g. through $(5,1,3)$

// $i + 4j - 2k$

$$\Rightarrow \begin{cases} x = 5+t \\ y = 1+4t \\ z = 3-2t \end{cases}$$

At what point intersect xy-plane

$z=0$

Find another pt.

symmetric equations of L.
$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

If none of a,b,c is 0,

If $a=0$, $b,c\neq 0$. $\quad x = x_0, \quad \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$.

Line segment from $\vec{r_0} = \langle x_0,y_0,z_0 \rangle$ to $\vec{r_1} = \langle x_1,y_1,z_1 \rangle$

$$\vec{r} = \vec{r_0} + t\vec{v}$$

$$t=1 \Rightarrow \vec{r_0} + \vec{v} = \vec{r_1}$$

$\Rightarrow$ vector equation $\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}, \quad 0 \le t \le 1$

skew lines do not intersect, not parallel.

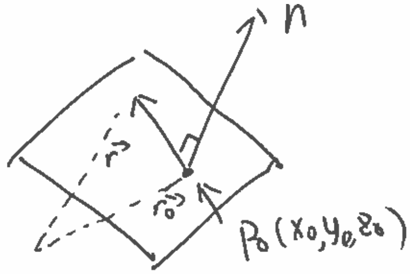
t, s solve the system $\Rightarrow$ no solut

• vectors not parallel

L_1: $x = 1+t$, $y = -2+3t$, $z = 4-t$

L_2: $x = 2s$, $y = 3+s$, $z = -3+4s$

Planes.



A plane is determined by a point P_0 and a vector $\vec{n}$ (orthogonal to the plane) called normal vector.

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \iff \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

$\vec{r} = \langle x, y, z \rangle$, $\vec{n} = \langle a, b, c \rangle$

$$\Downarrow$$

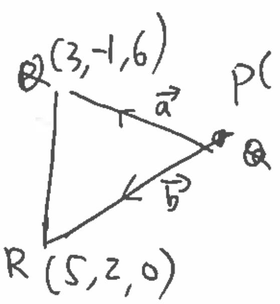
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

called scalar equation of the plane $\left\{ \begin{array}{l} \text{through } P_0 \\ \text{w/ } \vec{n} \text{ (normal vect)} \end{array} \right.$

$$\Rightarrow ax + by + cz + d = 0$$ linear equation.

$$d = -(ax_0 + by_0 + cz_0)$$

Oct 5



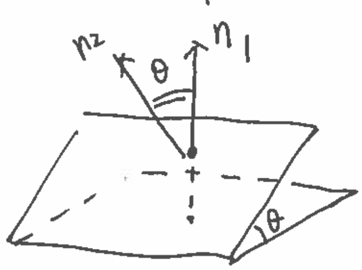
$\vec{a} = \langle 2, -4, 4 \rangle$

$\vec{b} = \langle 4, -1, -2 \rangle$

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12i + 20j + 14k$$

$$\Rightarrow 12(x-1) + 20(y-3) + 14(z-2) = 0.$$

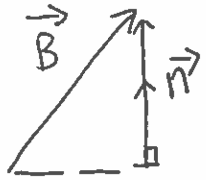
angle between 2 planes



$$\Rightarrow \cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

10, Oct 8.

Correction.



$$D = \frac{|\vec n \cdot \vec B|}{|\vec n|} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

HW. P880. graph by hand.

- Vector-valued functions: a function whose domain is a set of real numbers and whose range is a set of vectors.

Ex1: vectors $\vec v(t) = \langle 3-t,\ 2+3t,\ -1+5t \rangle = (3-t)\vec i + (2+3t)\vec j + (-1+5t)\vec k$

$t \longmapsto \vec v(t)$ vectors depends on choice of t.

$3-t,\ 2+3t,\ -1+5t$ are component functions of $\vec v(t)$

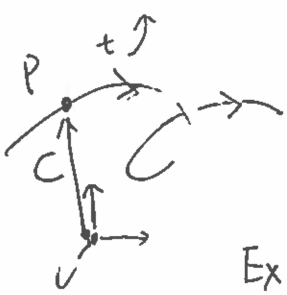
Ex2. $\vec r(t) = \langle 1, \sqrt t, e^t \rangle$

$\vec r(t) = \langle f(t), g(t), h(t) \rangle$

- Limits $\lim\limits_{t\to a} \vec r(t) = \langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t) \rangle$

$\vec r(t)$ is continuous at a if $\vec r(a) = \lim\limits_{t\to a} \vec r(t)$.

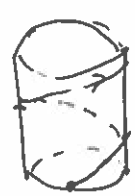
- Curves in space. e.g $\vec v(t)$ as above



In general C has parametric equations
$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$
parameter

$\vec r(t) = \langle f(t), g(t), h(t) \rangle$, $\vec r(t)$ is the position vector of a point on the curve.

Ex. helix



$\vec r(t) = \cos t\, \vec i + \sin t\, \vec j + t \cdot \vec k$.

Ex. Vector function
$$\begin{cases} \text{cylinder} & x^2+y^2=1 \\ \text{plane} & y+z=2 \end{cases}$$

$\vec v(\theta) = \langle \cos\theta, \sin\theta, 2-\sin\theta \rangle$

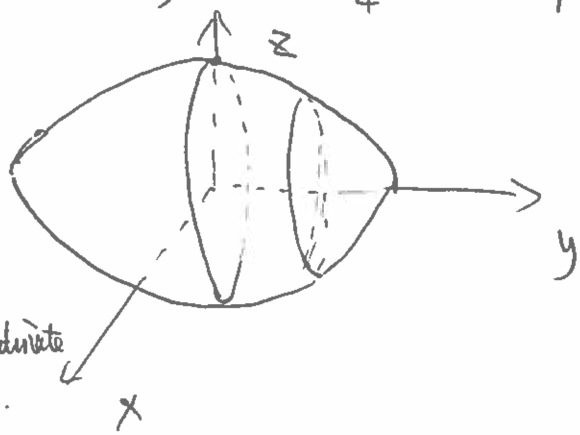
$0 \le \theta \le 2\pi$.

parametrization equation

- Use computer to draw space curves

Ex. ellipsoid $\quad x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$

$-3\le y\le 3$, $\quad y=k \Rightarrow x^2+\frac{z^2}{4}=1-\frac{k^2}{9}$



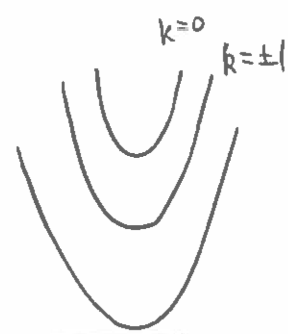
$x=0 \Rightarrow \frac{y^2}{9}+\frac{z^2}{4}=1$

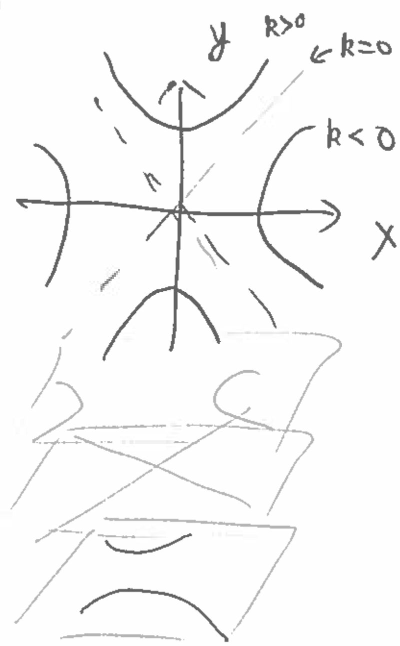
trace:
Intersections of surfaces and planes parallel to the coordinate plane.

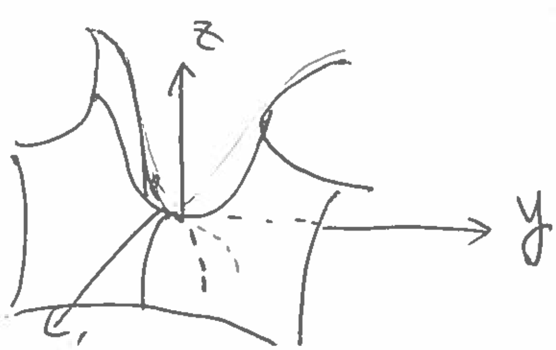
Ex. hyperbolic paraboloid.

$$z=y^2-x^2.$$

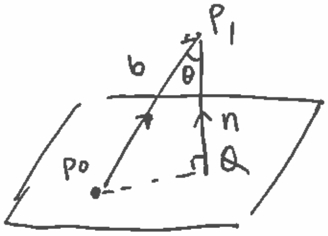
$x=k \quad z=y^2-k^2$; $\quad y=k \Rightarrow z=-x^2+k^2$; $\quad z=k \Rightarrow y^2-x^2=k$







Distance • from a point ~~Q~~ (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$



$$D = \left|\mathrm{Comp}_{\vec{n}}\vec{b}\right| = \frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

$\vec{b} = \langle x_1-x_0,\ y_1-y_0,\ z_1-z_0\rangle$

$\vec{n} \nparallel \overrightarrow{QP_1}$

d • between two parallel planes.

$10x+2y-2z=5$ a point $(\frac{1}{2},0,0)$

$5x+y-z=1$

• $\langle 10,2,-2\rangle$ parallel $\langle 5,1,-1\rangle$

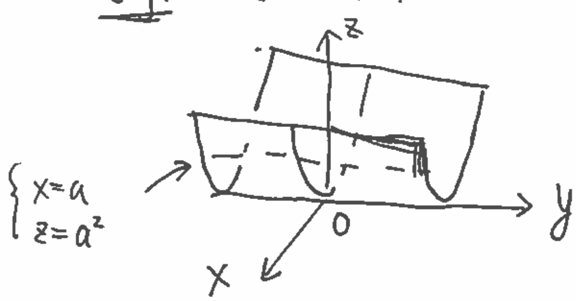
$$\Rightarrow D = \frac{|\frac{5}{2}-1|}{\sqrt{5^2+1^2+(-1)^2}} = \cdots$$

§12.6. Cylinders and Quadratic Surfaces.

surfaces: sphere, plane, ...

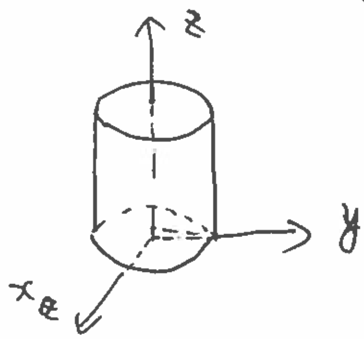
• A cylinder is a surface consists of all lines (rulings) that a parallel to a give line and pass through a given plane curve ↖ Trace

• e.g. $z=x^2$. parabolic cylinder



• passing through
  • $y=0,\ z=x^2$ is a parabola (a curve)
• parallel to $x=z=0$ (a line)
  $\langle 0,1,0\rangle$ is the direction vector

• e.g. $x^2+y^2=1$



• passing through $x^2+y^2=1,\ z=0$
• parallel to $x=y=0$.

Oct 8

• A quadratic surface ~~is of the form~~ is the graph of an equation $Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0$

$A, B, \cdots, J$ are numbers.

translation + rotation $\Rightarrow$ 2 standard forms $\begin{cases} Ax^2+By^2+Cz^2+J=0 & \text{Ellipsoid, cone, hyperboloid} \\ Ax^2+By^2+Iz=0 & \text{Elliptic Paraboloid, hyperbolic paraboloid} \end{cases}$

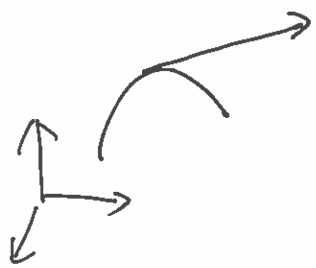
Oct 10

Vector function

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

tangent vector $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$ exists if $f'(t), g'(t), h'(t)$ exist and $\vec{r}'(t) \neq 0$.

$\Rightarrow$ unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|r'(t)|}$.



E.g. Find the parametric equations of tangent line to

$$\begin{cases} x = t^3 - 1 \\ y = t^4 + 1 \\ z = t^2 \end{cases} \quad \text{at } (7, 17, 4)$$

$\Rightarrow \vec{r}'(t) = \langle 3t^2, 4t^3, 2t \rangle$

$t = 2$

$\Rightarrow$ tangent line $\begin{cases} x = 12t + 7 \\ y = 32t + 17 \\ z = 4t + 4 \end{cases}$

Differentiation rules

$\vec{u}(t), \vec{v}(t)$ differentiable vect funcs.

c scalar, $f(t)$ real-valued function

- $\frac{d}{dt}(\vec{u} + \vec{v}) = \vec{u}' + \vec{v}'$
- $\frac{d}{dt}(c\vec{u}) = c\vec{u}'$
- $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
- $\frac{d}{dt}(\vec{u} \cdot \vec{v}) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$
- $\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
- $\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\vec{u}'(f(t))$. chain

$$(a_2b_3 - a_3b_2)' = a_2'b_3 + a_2b_3' - a_3'b_2 - a_3b_2'$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex. $|\vec{r}(t)| = c \Rightarrow \vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2$

$\Rightarrow 2\vec{r}'(t) \cdot \vec{r}(t) = 0 \Rightarrow \vec{r}'(t) \perp \vec{r}(t)$

Oct 10.

Integral

The definite Integrals of a continuous vector valued function $\vec{r}(t)$ is

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle.$$

If $\vec{R}(t)$ s.t. $\vec{R}'(t) = \vec{r}(t)$

$$\Rightarrow \int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big]_a^b = \vec{R}(b) - \vec{R}(a)$$

Ex. antiderivative $\int \vec{r}(t)dt$ is vector-valued functions

Ex $\vec{r}(t) = \langle 2\cos t, \sin t, 2t \rangle$

$$\Rightarrow \int \vec{r}(t) = \langle 2\sin t, -\cos t, t^2 \rangle + \vec{C}$$

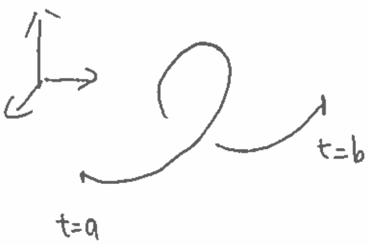
a constant vector

$$\Rightarrow \int_0^{\pi/2} \vec{r}(t)\,dt = \left\langle 2(\sin\tfrac{\pi}{2} - \sin 0), -\cos\tfrac{\pi}{2} + \cos 0, \left(\tfrac{\pi}{2}\right)^2 - 0^2 \right\rangle$$

$$= \left\langle 2, 1, \frac{\pi^2}{4} \right\rangle$$

Oct 12

Arc length Formula for a space curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$ ← differentiable functions



arc length

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

$$= \int_a^b |\vec{r}\,'(t)|\, dt.$$

integrate the speed function $|r'(t)|$

• $r'(t)$ is the velocity function

Ex. arc of the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ from $(1,0,0)$ to $(1,0,2\pi)$

i.e. $t=0$ to $t=2\pi$

$$r'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

• Arc length does not depend on the parametric eqs. (chain rule)

e.g. $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$, $1 \le t \le 2$

$\vec{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle$, $0 \le u \le \ln 2$ → same arc.

$t = e^u$ + chain rule $\quad L = \int_1^2 \sqrt{1 + (2t)^2 + (3t^2)^2}\, dt$

• Arc length function $s(t)$, $a \le t \le b$

is the arc length of C from a to t.

$$s(t) = \int_0^t |\vec{r}\,'(u)|\, du \Longrightarrow \frac{d\,s(t)}{dt} = |\vec{r}\,'(t)| \quad (*)$$

• parametrize a curve with respect to its arc length

$\frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{2}$ for $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$$\Rightarrow s(t) = \sqrt{2}\, t \quad \Rightarrow t(s) = \frac{s}{\sqrt{2}}$$

$$\Rightarrow \vec{r}(t(s)) = \left\langle \cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle; \quad |\vec{r}\,'(s)| = 1$$

because

chain rule

$$|\vec{r}\,'(s)| = \left| \frac{d}{ds} \vec{r}(t(s)) \right| = \left| \frac{d}{dt} \vec{r}(t) \frac{dt}{ds} \right|$$

$$= \left| \frac{d}{dt} \vec{r}(t) \right| \left(\frac{dt}{ds} \right) \overset{\text{use } (*)}{=\!=} 1$$