3/2/2019 MATH 551 REVIEW 1

6 problems to discuss today

Midterm has 4 questions

1) Let $F(x) = x + \ln x = 0$ then

a) Perform one step of Newton's method with $x_0 = 2$ [8 pts]

b) Perform one step of Secant method with $x_0 = 2$, $x_1 = 1$ [8 pts]

c) When should we use Secant method instead of Newton's method? [4 pts]

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solution:

a) $$X_{k+1} = X_k - \frac{F(x_k)}{F'(x_k)}, \quad k \geq 0$$

$$F'(x) = 1 + \frac{1}{x}$$

then for $k=0$

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

$$" = 2 - \frac{2 + \ln 2}{3/2}$$

$$\boxed{x_1 = \frac{1 - \ln 2}{3/2}} \quad (1)$$

b) $$X_{k+1} = X_k - \frac{F(x_k)(x_k - x_{k-1})}{F(x_k) - F(x_{k-1})}, \quad k \geq 1$$

$$x_2 = x_1 - \frac{F(x_1)(x_1 - x_0)}{F(x_1) - F(x_0)}$$

$$" = 1 - \frac{(-1)}{1 - 2 - \ln 2}$$

$$" = 1 - \frac{1}{+1 + \ln 2}$$

$$" = \frac{\cancel{1} + \ln 2 - \cancel{1}}{1 + \ln 2}$$

$$\boxed{x_2 = \frac{\ln 2}{1 + \ln 2}} \quad (2)$$

c) • the derivative might not be available

• $f'(x)$ is costly to evaluate

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2) Consider the functions [25 pts]

$$\begin{cases} f(x,y) = 3x^2 + 4y^2 - 1 = 0 \\ g(x,y) = y^3 - 8x^3 - 1 = 0 \end{cases}$$

Perform one step of Newton's method with initial guesses $(x_0, y_0) = (1, 1)$

HINT: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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solution:

$$\boxed{\vec{X}^{(k+1)} = \vec{X}^{(k)} - J^{-1}(\vec{X}^{(k)}) \vec{F}(\vec{X}^{(k)}), \quad k \geq 0} \quad (1)$$

$$\begin{bmatrix} X_{k+1} \\ Y_{k+1} \end{bmatrix} = \begin{bmatrix} X_k \\ Y_k \end{bmatrix} - J^{-1}(X_k, Y_k) \begin{bmatrix} f(X_k, Y_k) \\ g(X_k, Y_k) \end{bmatrix}$$

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 6x & 8y \\ -24x^2 & 3y^2 \end{bmatrix}$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - J^{-1}(x_0, y_0) \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix}$$

$$J(1,1) = \begin{bmatrix} 6 & 8 \\ -24 & 3 \end{bmatrix}, \quad J^{-1}(1,1) = \frac{1}{210} \begin{bmatrix} 3 & -8 \\ 24 & 6 \end{bmatrix}$$

$18 + 8 \cdot 24$

$18 + 192 = 210$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{210} \begin{bmatrix} 3 & -8 \\ 24 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{210} \begin{bmatrix} 82 \\ 96 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{210}\begin{bmatrix} 210-82 \\ 210-96 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{210}\begin{bmatrix} 128 \\ 114 \end{bmatrix}$$

Extra: $f(x,y) = 3x + y = 0$
$g(x,y) = x^2 + 5y - 2 = 0$
$(x_0, y_0) = (-2, 1)$

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3) Convert $x^2 - 5 = 0$ to the fixed-point problem:

$$\underbrace{x = x + c(x^2-5)}_{f(x)} \quad / \quad c \in \mathbb{R} - \{0\}$$

Determine the possible values of c to ensure convergence of

$$x_{n+1} = x_n + c(x_n^2 - 5) \text{ to } x^* = \sqrt{5}$$

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solution:

$$x = x + c(x^2-5) \Rightarrow c(x^2-5) = 0 \nearrow \cancel{c=0}, \searrow x^2-5=0$$

$$x^2 - 5 = 0 \Rightarrow e^x = x^2 - 5 + e^x \Rightarrow$$

$$x = \underbrace{\ln(x^2 - 5 + e^x)}_{g(x)}$$

$$x^2 - 5 + \cos(x) = \cos(x) \Rightarrow$$

$$x = \cos^{-1}\left[x^2 - 5 + \cos(x)\right],$$

$$g(x) = \cos^{-1}\left[x^2 - 5 + \cos(x)\right]$$

$|g'(x)| < 1$, $x \in (a,b)$ s.t. $x^* \in (a,b)$, $[x^* - \delta, x^* + \delta]$, $\delta << 1$

$$g(x) = x + c(x^2 - 5),$$

$$g'(x) = 1 + 2cx$$

$$g'(x^*) = 1 + 2c\sqrt{5} \rightarrow$$

$$|g'(x^*)| < 1$$

$$|1 + 2c\sqrt{5}| < 1$$

$$-1 < 1 + 2c\sqrt{5} < 1$$

$$0 < 2 + 2c\sqrt{5} < 2$$

$$\boxed{0 < 1 + c\sqrt{5} < 1}$$

$$\frac{-1}{\sqrt{5}} < c < 0$$

$$1 + c\sqrt{5} = 0 \Rightarrow c = -\frac{1}{\sqrt{5}}$$

c	$-\infty$	$-\frac{1}{\sqrt{5}}$	0	$+\infty$
$1 + c\sqrt{5}$	$-$			

$$\Rightarrow \frac{-1}{\sqrt{5}} < c < 0$$

$g'(x^*) \neq 0 \Rightarrow$ converges linearly

$g'(x^*) = 0 \Rightarrow$ converges at least quadratically

- $g''(x^*) \neq 0$ quadratically
- $g''(x^*) = 0$ at least cubically
  - $g'''(x^*)$
    - $g'''(x^*) \neq 0$ cubically
    - $g'''(x^*) = 0$ at least quintically

$c = -\frac{1}{\sqrt{5}}$ $\quad g'(x^*) = 0$

$g''(x) = 2c \quad g''(x^*) = -\frac{1}{\sqrt{5}} \neq 0$

$$c = -\frac{1}{2\sqrt{5}}$$

4) Which of the following iterations will converge to x^* provided that x_0 is chosen sufficiently close to x^*

a) $x_{k+1} = \dfrac{15x_k^2 - 24x_k + 13}{4x_k}$, $x^* = 1$

b) $x_{k+1} = \dfrac{3}{4}x_k + \dfrac{1}{x_k^3}$, $x^* = \sqrt{2}$

Determine the order of convergence

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solution:

a) $g(x) = \dfrac{15x^2 - 24x + 13}{4x}$

$g(1) = \dfrac{15 - 24 + 13}{4} = 1$

$g'(x) = \dfrac{(30x-24)4x - 4(15x^2 - 24x + 13)}{16x^2}$

$g'(1) = \dfrac{24-16}{16} \Rightarrow g'(1) = \frac{1}{2}$,

$|g'(1)| < 1$

linearly !

b) $g(x) = \frac{3}{4}x + \frac{1}{x^3}$

$g(\sqrt{2}) = \frac{3}{4}\sqrt{2} + \frac{1}{(\sqrt{2})^3}$

$\quad = \frac{3}{4}\sqrt{2} + \frac{1}{2^{3/2}}$

$\quad = 3\, 2^{-2} 2^{1/2} + 2^{-3/2}$

$\quad = 3\, 2^{-3/2} + 2^{-3/2}$

$\quad = 4\, 2^{-3/2}$

$\quad = 2^2\, 2^{-3/2}$

$\quad = 2^{1/2}$

$\quad = \sqrt{2}$

$g'(x) = \frac{3}{4} - 3x^{-4}$

$g'(\sqrt{2}) = \frac{3}{4} - 3(2^{1/2})^{-4} \Rightarrow$

$g'(x^*) = \frac{3}{4} - 3\, 2^{-2} = 0$

$g''(x) = 12\, x^{-5}$

$g''(x^*) = \dfrac{12}{2^{5/2}} \neq 0$

quadratically !

5) Algorithm: Bisection method

A function fun and three parameters boundary A, boundary B, and Error tolerance are given.

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02. If fun(boundaryA) x fun(boundary B) < 0 then
03..... Display "something wrong with the input"
04.. else
05...... ValueA = fun(boundary A)
06...... ValueB = fun(boundary A)
07...... While boundary B - boundary A > Error tolerance Do
08......     Midpoint = boundary A + boundary B
09......     ValueMidpoint = Fun(boundary A)
10......     if ValueMidpoint x ValueA > 0 then
11......         boundary B = Midpoint
12......     else
13......         boundary A = Midpoint
14......     end if
15...... end while
16......   solution = boundary A + boundary B
17......   return solution
18.  end if
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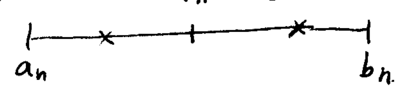
remember: $p_n = \dfrac{a_n + b_n}{2}$



- Find the mistakes [5 pts]
  - line 08: Midpoint = (boundary A + boundary B)/2
  - line 02: if fun(boundaryA) x fun(boundary B) > 0
  - line 06: Value B = fun (boundary B)
  - line 11: boundary A = Midpoint
  - line 13: boundary B = Midpoint
  - line 09: ValueMidpoint = fun (Midpoint)
  - line 16: Solution = (boundary A + boundary B)/2
  - INSERT BETWEEN LINES 11 AND 12: ValueA = ValueMidpoint
  - INSERT BETWEEN LINES 13 AND 14: ValueB = ValueMidpoint

- If fun = $x^2 - 3x - 10$, boundary A = 0, boundary B = 6, Error tolerance = 0.4, write the output at the corrected algorithm

Fun(0) x Fun(6) = $-10.8 < 0$
ValueA = -10
Value B = 8
MidPoint = 3,
ValueMidPoint = -10
boundary A = 3
Value A = -10

$\leftarrow$ Iteration 1

MidPoint $= 9/2$

ValueMidPoint $= \frac{81}{4} - \frac{27}{2} - 10$

$= \frac{81 - 54 - 40}{4}$

$= \frac{-13}{4} < 0$

boundary A $= 9/2$

Value A $= \frac{-13}{4}$

← iteration 2

boundary B − boundary A

$= 6 - 9/2$

$= 3/2 > 0.4$

midpoint $= \frac{9/2 + 6}{2}$

$= \frac{21}{4}$

Value Midpoint

$= \frac{(21)^2}{16} - 3\,\frac{21}{4} - 10$

$= \frac{441 - 12\; 21 - 160}{16}$

$= \frac{441 - 252 - 160}{16}$

$= 29/16$

boundary B $= \frac{21}{4}$

Value B $= \frac{29}{16}$

← iteration 3

boundary A $= 9/2$

boundary B $= 21/4$

Midpoint $= 39/8$

ValueMidpoint

$= \frac{(39)^2}{64} - 3\,\frac{39}{8} - 10$

$= \frac{(39)^2 - 24\; 39 - 640}{64}$

$= -55/64$

boundary A $= 39/8$

Value A $= -55/64$

← iteration 4

$\frac{21}{4} - \frac{39}{8}$

$\frac{42 - 39}{8}$

$\frac{3}{8}$

0.375 ✓

solution $= \frac{39/8 + 21/4}{2}$

$'' = \frac{39 + 42}{16}$

$'' = \boxed{\frac{81}{16}}$

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$$

$$b_n - a_n = \frac{1}{2^{n-1}}(b - a)$$

$$b_{n+1} - a_{n+1} = \frac{1}{2^n}(b - a)$$

$$\boxed{\left| x^* - p_n \right| \le \frac{1}{2^n}(b - a)}, \quad n \to \infty, \quad \lim_{n\to\infty} p_n = x^*$$

$$\left| x^* - p_n \right| \le \varepsilon$$

$$n \ge \frac{\log\left(\frac{b-a}{\varepsilon}\right)}{\log 2}$$