

- A "Mole" of a substance = its molecular weight M_w in grams.
- kmole or kg-mole = its molecular weight in kilograms.
- lb-mole = its molecular weight in lb (pound mass).
- c = Concentration of a substance in a solution = moles per liter (mole/l) = (kmole/m³)
- Hence, by the above definition of concentration, number of moles n in solution volume V having concentration c is:

$$n = Vc \quad (11 - 23)$$

Where:

c = Concentration of solution (kmole/m³)

V = Volume of the solution (m³)

n = Number of moles in V (kmole)

- When there is a flow, a differential volume dV is moved in dt seconds. This delivers the following number of moles in time dt :

$$dn = cdV$$

- Divide the above by time differential dt and recall definition of volume flow rate $f \equiv \frac{dV}{dt}$:

$$\dot{n} = c\dot{V}; f \equiv \dot{V} \Rightarrow$$

$$\dot{n} = cf \quad (11 - 24)$$

- Where:

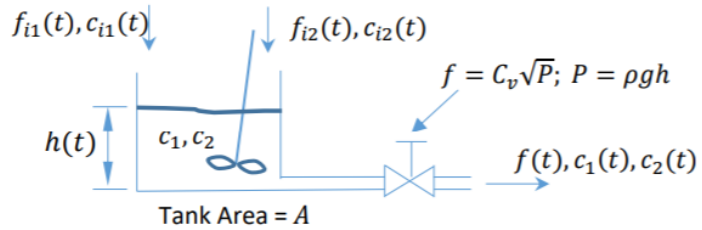
\dot{n} = The rate at which moles are delivered by the flow (kmole/s)

f = Volume flow rate of the solution (m³/s).

c = Concentration of the dissolved substance in the flow (kmole/m³)

Example - Two Substances Mixed in a Liquid Tank

- Consider the mixing tank shown in the figure on right, it mixes two substances at room temperature.
- The volume flow rates of substances 1 and 2 entering the tank are: $f_{i1}(t), f_{i2}(t)$
- The concentrations of the substances in the inflows are: $c_{i1}(t), c_{i2}(t)$.
- The concentrations of the substances inside the tanks are $c_1(t), c_2(t)$.
- The output flow restrictor is a valve with valve coefficient C_v .
- The output volume flow rate is $f(t)$.
- Find system state space equations; note that system is nonlinear.



Answer:

- The system has 3 state variables: h, c_1, c_2 (as their derivatives show up in the equations).
- System has 4 inputs: $f_{i1}, c_{i1}, f_{i2}, c_{i2}$.
- System has 3 outputs: f, c_1, c_2 .
- Hence state, input and output vectors are:

$$\mathbf{x} = \begin{bmatrix} h \\ c_1 \\ c_2 \end{bmatrix}; \mathbf{u} = \begin{bmatrix} f_{i1} \\ c_{i1} \\ f_{i2} \\ c_{i2} \end{bmatrix}; \mathbf{y} = \begin{bmatrix} f \\ c_1 \\ c_2 \end{bmatrix}$$

- In Handout 7 on page 2, the differential equation of the height of a liquid inside of the tank was derived:

Height of Liquid:

$$\frac{dh}{dt} = \frac{1}{A} \sum (\text{Volume Flow Rates in}) - \frac{1}{A} \sum (\text{Volume Flow Rates out}) = \frac{f_{i1} + f_{i2} - f}{A}$$

- The outflow is given by the valve equation (specific gravity of liquid $G_f=1$):

$$f = C_v \sqrt{P}; P = \rho gh \Rightarrow f = C_v \sqrt{\rho gh}$$

- Note that pressure drop across the valve is: $\Delta P = P - 0$ as the flow discharges at atmospheric pressure.
- Substituting in expression for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{-C_v \sqrt{\rho gh} + f_{i1} + f_{i2}}{A} \quad (11 - 25)$$

Where A = Cross sectional area of the tank (constant)

Mole Balance:

- Differential mole balance equation for the tank where n is moles of a substance is:

$$dn_{in} - dn_{out} = dn_{\text{stored in tank}}$$

- Divide the above by dt :

$$\dot{n}_{in} - \dot{n}_{out} = \left. \frac{dn}{dt} \right|_{\text{Inside Tank}}$$

- Use the expression $\dot{n} = cf$ (equation 11 – 24) in the above:

$$c_i f_i - c_o f_o = \left. \frac{dn}{dt} \right|_{\text{Inside Tank}} \quad (11 - 26)$$

- Write equation (11 – 23) for moles in the tank where V = volume of liquid in tank:

$$n = cV$$

- Now differentiate the above result with respect to time to find the right-hand side of (11 – 26):

$$\frac{d}{dt}(n = cV) \Rightarrow \frac{dn}{dt} = c \frac{dV}{dt} + V \frac{dc}{dt}$$

- Note that both c and h are functions of time. Substitute the above for the right-hand side of (11 – 26):

$$c_i f_i - c_o f_o = c \frac{dV}{dt} + V \frac{dc}{dt}$$

- Note that in the above the dissolved substance concentration in the exit flow is the same as the concentration inside the tank: $c_o = c$ for each substance; also f_o is the same as f in the problem statement. For substances 1 and 2, the above equation becomes:

$$\begin{cases} c_{i1} f_{i1} - c_1 f = c_1 \frac{dV}{dt} + V \frac{dc_1}{dt} \\ c_{i2} f_{i2} - c_2 f = c_2 \frac{dV}{dt} + V \frac{dc_2}{dt} \end{cases}$$

- Now, the rate of change of liquid volume in tank dV/dt is given by (See Handout 7 page 2):

$$\frac{dV}{dt} = (\text{Sum of Volume Flow Rates into the Tank}) - (\text{Sum of Volume Flow Rates out of the Tank}) \Rightarrow$$

$$\frac{dV}{dt} = f_{i1} + f_{i2} - f$$

- Substitute for dV/dt in the previous equation; also note that $V = Ah$:

$$\begin{cases} c_{i1} f_{i1} - c_1 f = (f_{i1} + f_{i2} - f)c_1 + Ah \frac{dc_1}{dt} \\ c_{i2} f_{i2} - c_2 f = c_2 (f_{i1} + f_{i2} - f) + Ah \frac{dc_2}{dt} \end{cases}$$

- Expand, simplify and solve for dc_1/dt and dc_2/dt :

$$\begin{cases} \frac{dc_1}{dt} = \frac{c_{i1} f_{i1} - c_1 f - c_1 f_{i1} - c_1 f_{i2} + c_1 f}{Ah} \\ \frac{dc_2}{dt} = \frac{c_{i2} f_{i2} - c_2 f - c_2 f_{i1} - c_2 f_{i2} + c_2 f}{Ah} \end{cases}$$

$$\begin{cases} \frac{dc_1}{dt} = -\frac{f_{i1} + f_{i2}}{Ah} c_1 + \frac{f_{i1}}{Ah} c_{i1} \\ \frac{dc_2}{dt} = -\frac{f_{i1} + f_{i2}}{Ah} c_2 + \frac{f_{i2}}{Ah} c_{i2} \end{cases}$$

- Combine the above with equation (11 – 25) into vector form to find the (nonlinear) state space equations; recall $f = C_v \sqrt{\rho g h}$:

$$\frac{dx}{dt} = f(x, u) \Rightarrow \frac{d}{dt} \begin{bmatrix} h \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -C_v \sqrt{\rho g h} + f_{i1} + f_{i2} \\ A \\ -\frac{f_{i1} + f_{i2}}{Ah} c_1 + \frac{f_{i1}}{Ah} c_{i1} \\ -\frac{f_{i1} + f_{i2}}{Ah} c_2 + \frac{f_{i2}}{Ah} c_{i2} \end{bmatrix}; \quad y = g(x, u) \Rightarrow \begin{bmatrix} f \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} C_v \sqrt{\rho g h} \\ c_1 \\ c_2 \end{bmatrix} \quad (11 - 27)$$

- In the expression for the output equation, c_1 and c_2 are repeated on both sides. On the left c_1, c_2 exist because they are system outputs and hence are included in output vector y . On the right c_1, c_2 are repeated because they are also state variables. (Recall the form of $g(x, u)$; it should be shown as a function of state variables and system inputs.)

Steady State Operation:

- Given system is operated at steady state at nominal input values: $\bar{f}_{i1}; \bar{c}_{i1}; \bar{f}_{i2}; \bar{c}_{i2}$ find the steady state values of the state variables and system output.
- To do so, set the derivatives equal to 0 as at steady state the state variables are constants:

$$\frac{d}{dt} \begin{bmatrix} h \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -C_v \sqrt{\rho g h} + \bar{f}_{i1} + \bar{f}_{i2} \\ A \\ -\frac{\bar{f}_{i1} + \bar{f}_{i2}}{Ah} \bar{c}_1 + \frac{\bar{f}_{i1}}{Ah} \bar{c}_{i1} \\ -\frac{\bar{f}_{i1} + \bar{f}_{i2}}{Ah} \bar{c}_2 + \frac{\bar{f}_{i2}}{Ah} \bar{c}_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -C_v \sqrt{\rho g h} + \bar{f}_{i1} + \bar{f}_{i2} = 0 \\ -(\bar{f}_{i1} + \bar{f}_{i2}) \bar{c}_1 + \bar{f}_{i1} \bar{c}_{i1} = 0 \\ -(\bar{f}_{i1} + \bar{f}_{i2}) \bar{c}_2 + \bar{f}_{i2} \bar{c}_{i2} = 0 \end{cases}$$

$$\begin{cases} \bar{h} = \frac{(\bar{f}_{i1} + \bar{f}_{i2})^2}{\rho g C_v^2} \\ \bar{c}_1 = \frac{\bar{f}_{i1} \bar{c}_{i1}}{(\bar{f}_{i1} + \bar{f}_{i2})} \\ \bar{c}_2 = \frac{\bar{f}_{i2} \bar{c}_{i2}}{(\bar{f}_{i1} + \bar{f}_{i2})} \end{cases} \quad (11 - 28A)$$

- The outflow steady state value is:

$$\bar{f} = \bar{f}_{i1} + \bar{f}_{i2} \quad (11 - 28B)$$

- Now you can Linearize the system at steady state point. See Handout 11A for Linearization of above.

Linearization:

- Linearize system at the following steady state operating point; write the final state space system equation in terms of deviation variables:

$$\bar{f}_{i1} = \bar{f}_{i2} = 0.005; \bar{c}_{i1} = \bar{c}_{i2} = 1.0 \\ C_v = 10^{-4}; A = 1; \rho = 1000; g = 10$$

- Follow Handout 9.
- First step is to define all symbols by `syms` command.
- Then define the **x**, **u**, **y** column vectors using the defined symbols.
- Finally, define the column vector functions **f(x,u)**, **g(x,u)**.
- See MATLAB code below, read comments for explanations:

Symbols and Definitions:

```
syms h c1 c2                % State variables.
syms fi1 ci1 fi2 ci2        % System inputs.
syms f                       % Output.
syms a r g cv                % Parameters, r = rho (density)

x = [h; c1; c2]             % State vector.
u = [fi1; ci1; fi2; ci2]    % Input vector.
y = [f; c1; c2]             % Output vector.

fxu = [(-cv * sqrt(r*g*h) + fi1 + fi2)/a; % f(x,u) vector of functions.
       -(fi1+fi2)*c1/(a*h) + fi1*ci1/(a*h);
       -(fi1+fi2)*c2/(a*h) + fi2*ci2/(a*h)]

f = cv * sqrt(r*g*h)        % Outflow equation.
gxu = [f; c1; c2]          % g(x,u).
```

x =

h
c1
c2

u =

fi1
ci1
fi2
ci2

y =

f
c1
c2

fxu =

$$\begin{aligned} & (fi1 + fi2 - cv^*(g^*h^*r)^{(1/2)})/a \\ & (ci1*fi1)/(a^*h) - (c1*(fi1 + fi2))/(a^*h) \\ & (ci2*fi2)/(a^*h) - (c2*(fi1 + fi2))/(a^*h) \end{aligned}$$

f =

$$cv^*(g^*h^*r)^{(1/2)}$$

gxu =

$$\begin{aligned} & cv^*(g^*h^*r)^{(1/2)} \\ & \quad c1 \\ & \quad c2 \end{aligned}$$

A,B,C,D matrices:

```
AA = jacobian(fxu,x)
BB = jacobian(fxu,u)
CC = jacobian(gxu,x)
DD = jacobian(gxu,u)
```

AA =

$$\begin{bmatrix} -(cv^*g^*r)/(2*a^*(g^*h^*r)^{(1/2)}), & 0, & 0 \\ (c1*(fi1 + fi2))/(a^*h^2) - (ci1*fi1)/(a^*h^2), & -(fi1 + fi2)/(a^*h), & 0 \\ (c2*(fi1 + fi2))/(a^*h^2) - (ci2*fi2)/(a^*h^2), & 0, & -(fi1 + fi2)/(a^*h) \end{bmatrix}$$

BB =

$$\begin{bmatrix} 1/a, & 0, & 1/a, & 0 \\ ci1/(a^*h) - c1/(a^*h), & fi1/(a^*h), & -c1/(a^*h), & 0 \\ -c2/(a^*h), & 0, & ci2/(a^*h) - c2/(a^*h), & fi2/(a^*h) \end{bmatrix}$$

CC =

$$\begin{bmatrix} (cv^*g^*r)/(2^*(g^*h^*r)^{(1/2)}), & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix}$$

DD =

$$\begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}$$

Steady State Calculations:

```
% Steady state values of inputs:
fi1=0.005
ci1=1.0
fi2=.005
ci2=1.0

% System parameters:
a = 1
r = 1000
g = 10
cv = 1e-4

% Steady state values of state variables Handout 11 p14 eq. (11 - 28A):
h_ = (fi1+ fi2)^2/(r*g*cv^2)

c1_ = fi1* ci1 / (fi1+ fi2)
c2_ = fi2* ci2 / (fi1+ fi2)
```

```
fi1_ =
    0.0050
```

```
ci1_ =
    1
```

```
fi2_ =
    0.0050
```

```
ci2_ =
    1
```

```
a =
    1
```

```
r =
   1000
```

```
g =
    10
```

```
cv =
   1.0000e-04
```

```
h_ =
    1
```

```
c1_ =
    0.5000
```

```
c2_ =
    0.5000
```

Substitution of the operating point values in the A,B,C,D matrices.

```
A = subs(AA, {h, c1, c2, fi1, ci1, fi2, ci2}, ...
           {h_, c1_, c2_, fi1_, ci1_, fi2_, ci2_})

B = subs(BB, {h, c1, c2, fi1, ci1, fi2, ci2}, ...
           {h_, c1_, c2_, fi1_, ci1_, fi2_, ci2_})

C = subs(CC, {h, c1, c2, fi1, ci1, fi2, ci2}, ...
           {h_, c1_, c2_, fi1_, ci1_, fi2_, ci2_})

D = subs(DD, {h, c1, c2, fi1, ci1, fi2, ci2}, ...
           {h_, c1_, c2_, fi1_, ci1_, fi2_, ci2_})
```

```
A =
[ -(cv*g*r)/(2*a*(g*r)^(1/2)), 0, 0]
[ 0, -1/(100*a), 0]
[ 0, 0, -1/(100*a)]
```

```
B =
[ 1/a, 0, 1/a, 0]
[ 1/(2*a), 1/(200*a), -1/(2*a), 0]
[ -1/(2*a), 0, 1/(2*a), 1/(200*a)]
```

```
C =
[ (cv*g*r)/(2*(g*r)^(1/2)), 0, 0]
[ 0, 1, 0]
[ 0, 0, 1]
```

```
D =
[ 0, 0, 0, 0]
[ 0, 0, 0, 0]
[ 0, 0, 0, 0]
```

IMPORTANT: Below substitutes numerical values for any variable that wasn't substituted before such as: cv, a, r, g.

```
A = subs(A)
B = subs(B)
C = subs(C)
D = subs(D)
```

```
A =
[ -1/200, 0, 0]
[ 0, -1/100, 0]
[ 0, 0, -1/100]
```

B =
 $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1/2 & 1/200 & -1/2 & 0 \\ -1/2 & 0 & 1/2 & 1/200 \end{bmatrix}$

C =
 $\begin{bmatrix} 1/200 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D =
 $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Convert to double precision numbers (Final Answers):

A = double(A)
 B = double(B)
 C = double(C)
 D = double(D)

A =
 $\begin{bmatrix} -0.0050 & 0 & 0 \\ 0 & -0.0100 & 0 \\ 0 & 0 & -0.0100 \end{bmatrix}$

B =
 $\begin{bmatrix} 1.0000 & 0 & 1.0000 & 0 \\ 0.5000 & 0.0050 & -0.5000 & 0 \\ -0.5000 & 0 & 0.5000 & 0.0050 \end{bmatrix}$

C =
 $\begin{bmatrix} 0.0050 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}$

D =
 $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- The system state equations in terms of deviation variables are:

$$\frac{d}{dt} \begin{bmatrix} \delta h \\ \delta c_1 \\ \delta c_2 \end{bmatrix} = \begin{bmatrix} -0.005 & 0 & 0 \\ 0 & -0.01 & 0 \\ 0 & 0 & -0.01 \end{bmatrix} \begin{bmatrix} \delta h \\ \delta c_1 \\ \delta c_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0.5 & 0.005 & -0.5 & 0 \\ -0.5 & 0 & 0.5 & 0.005 \end{bmatrix} \begin{bmatrix} \delta f_{i1} \\ \delta c_{i1} \\ \delta f_{i2} \\ \delta c_{i2} \end{bmatrix}$$

- Output equation:

$$\begin{bmatrix} \delta f \\ \delta c_1 \\ \delta c_2 \end{bmatrix} = \begin{bmatrix} 0.005 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta h \\ \delta c_1 \\ \delta c_2 \end{bmatrix}$$